Ex. 2 Use the method of undetermined coefficients to solve

$$x' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} x + \begin{pmatrix} 1 \\ t \end{pmatrix}$$

Sol'n: $A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$, $\det(A - \lambda I) = \lambda^2 = 0$

$r_1 = r_2 = 0$, $\quad \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$x^{(1)} = \xi^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$x^{(2)} = \xi^{(1)} t e^{r_1 t} + \eta e^{r_1 t} = t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \eta$

$(A - r_1 I)\eta = \xi \quad \Rightarrow \quad \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow 4\eta_1 - 2\eta_2 = 1$. Choose $\eta_2 = 0$, $\eta_1 = \left(\frac{1}{4}\right)$

$x^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} t + \frac{1}{4} \\ 2t \end{pmatrix}$

Now $\vec{g} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$ so $\vec{g}$ is of polynomial degree 1.

Try $x_p = t^s$ (polynomial of degree 1) + rest lower order

$= t^s(\vec{a} + \vec{b}t)$ + lower order

$s \neq 0, 1$ since $1, t$ are in the kernel

So $s = 2$

Thus

$$x_p = \vec{a}_0 t^3 + \vec{a}_1 t^2 + \vec{a}_2 t + \vec{a}_3$$

$$x_p' = 3\vec{a}_0 t^2 + 2\vec{a}_1 t + \vec{a}_2$$

$$Ax_p = A\vec{a}_0 t^3 + A\vec{a}_1 t^2 + A\vec{a}_2 t + A\vec{a}_3$$

$x_p' = Ax_p + g \Rightarrow$

$$3\vec{a}_0 t^2 + 2\vec{a}_1 t + \vec{a}_2 = A\vec{a}_0 t^3 + A\vec{a}_1 t^2 + A\vec{a}_2 t + A\vec{a}_3 + \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} t$$

t^3: $A\vec{a}_0 = 0$ —— ①

t^2: $A\vec{a}_1 = 3\vec{a}_0$ —— ②

t: $A\vec{a}_2 + \begin{pmatrix}0\\1\end{pmatrix} = 2\vec{a}_1$ —— ③

1: $A\vec{a}_3 + \begin{pmatrix}1\\0\end{pmatrix} = \vec{a}_2$ —— ④

Eqn ①: $a_0 = k_0 \begin{pmatrix}1\\2\end{pmatrix}$, k_0 is to be determined

Eqn ②: $\begin{pmatrix}4 & -2\\8 & -4\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = 3k_0 \begin{pmatrix}1\\2\end{pmatrix} \Rightarrow 4a_1 - 2a_2 = 3k_0$

choose $a_2 = 0$, $a_1 = \frac{3}{4}k_0$

$$\vec{a}_1 = \begin{pmatrix}\frac{3}{4}k_0\\0\end{pmatrix} + k_1 \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}\frac{3}{4}k_0 + k_1\\2k_1\end{pmatrix}$$

Eqn ③: $\begin{pmatrix}4 & -2\\8 & -4\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = 2\vec{a}_1 - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{3}{2}k_0 + 2k_1\\4k_1 - 1\end{pmatrix}$ —— ⑤

To have a solution to (5), we need

$$2\left(\tfrac{3}{2}k_0 + 2k_1\right) = 4k_1 - 1 \quad \text{— (6)}$$

Next we choose a $\vec{a}_2$

$$\vec{a}_2 = \begin{pmatrix} \frac{1}{8}(4k_1-1) \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

we can choose $k_2 = 0$ because it leads to $X^{(2)}$

Eqn (3): $A a_3 = a_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{8}(4k_1-1) - 1 \\ 0 \end{pmatrix} \quad \text{— (7)}$$

(7) has a sol'n $\Rightarrow$ $\frac{1}{8}(4k_1-1) - 1 = 0$ — (8)

Thus $k_1 = \frac{9}{4}$ — (9)

Substituting into (6), we get

$$\tfrac{3}{2}k_0 + \tfrac{9}{2} = 4 \Rightarrow k_0 = -\tfrac{1}{3}$$

So we can choose

$$a_3 = 0, \quad a_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} \frac{3}{4}k_0 + \cancel{4k_1} \\ 2k_1 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{9}{2} \end{pmatrix}, \quad a_0 = -\tfrac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$x_p = \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} t^3 + \begin{pmatrix} 2 \\ \frac{9}{2} \end{pmatrix} t^2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t$$

and

$$X = X_p + c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t + \frac{1}{4} \\ 2t \end{pmatrix}$$

To check, let us use Method II:

$$\Psi(t)\, u' = g$$

$$\begin{pmatrix} 1 & t+\frac{1}{4} \\ 2 & 2t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}$$

$$u_1' + (t + \tfrac{1}{4})\, u_2' = 1$$

$$2u_1' + 2t\, u_2' = t$$

$$\Rightarrow \quad \tfrac{1}{4} u_2' = 1 - \tfrac{t}{2} \quad \Rightarrow \quad u_2' = 4 - 2t \quad \Rightarrow \quad u_2 = 4t - t^2$$

$$u_1' = \tfrac{t}{2} - (4-2t)t = -\tfrac{7}{2} t + 2t^2$$

$$u_1 = -\tfrac{7}{4} t^2 + \tfrac{2}{3} t^3$$

$$x_p = \Psi(t)\, u = \begin{pmatrix} 1 & t+\frac{1}{4} \\ 2 & 2t \end{pmatrix} \begin{pmatrix} -\frac{7}{4}t^2 + \frac{2}{3}t^3 \\ 4t - t^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{7}{4}t^2 + \frac{2}{3}t^3 + (t+\frac{1}{4})(4t-t^2) \\ -\frac{7}{2}t^2 + \frac{4}{3}t^3 + 2t(4t-t^2) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3}t^3 + 2t^2 + t \\ \\ -\frac{2}{3}t^3 + \frac{9}{2}t^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} t^3 + \begin{pmatrix} 2 \\ \frac{9}{2} \end{pmatrix} t^2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t$$