MATH 256 : 103

Solution of assignment 1

1.3 4. order = 1, nonlinear

6 order = 3, linear

18. Putting $y = e^{rt}$ gives

$$\cancel{r^3 = 3y^2 + 2y}$$

$$r^3 - 3r^2 + 2r = 0$$

$$r(r-1)(r-2) = 0$$

$$r = 0,\ 1 \text{ or } 2. \qquad \square$$

2.1 16. $I.F. = e^{\int \frac{2}{t}} = t^2$

$$t^2 y' + 2ty = \cos t$$

$$(t^2 y)' = \cos t$$

$$t^2 y = \sin t + C$$

$$y = \frac{\sin t + C}{t^2}$$

Using $y(\pi) = 0$,

$$0 = \frac{0 + C}{\pi^2}$$

$$C = 0$$

$$y = \frac{\sin t}{t^2} \qquad \square$$

18. $$ty' + 2y = \sin t$$

$$y' + \frac{2}{t}y = \frac{1}{t}\sin t$$

As in Q16, I.F. $= t^2$

$$(t^2y)' = t^{\cancel{2}}\sin t$$

$$t^2y = \overset{-t\cos t \ + \sin t}{\cancel{-t^2\cos t + 2t\sin t + 2\cos t}} + C$$

$$y = -\frac{\cos t}{t} + \frac{\cancel{2}\sin t}{t^2} \cancel{+\frac{2\cos t}{t^2}} + \frac{C}{t^2}$$

Using $y(\frac{\pi}{2}) = 1$,

$$1 = 0 + \frac{\cancel{2}1}{(\frac{\pi}{2})^2} + 0 + \frac{C}{(\frac{\pi}{2})^2}$$

$$C = \cancel{2}\frac{\pi^2}{4} - \cancel{2}1$$

$$\therefore \quad y = -\frac{\cos t}{t} + \frac{\cancel{2}\sin t}{t^2} \cancel{+\frac{2\cos t}{t^2}} + \frac{\frac{\pi^2}{4} - \cancel{2}1}{t^2} \quad \square$$

19. $$t^3y' + 4t^2y = e^{-t}$$

$$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$$

$$\text{I.F.} = e^{\int\frac{4}{t}} = t^4$$

$$(t^4y)' = te^{-t}$$

$$t^4y = -(t+1)e^{-t} + C$$

Since $y(-1) = 0$, $0 = 0 + C$.

$$\therefore \quad y = -\frac{t+1}{t^4}e^{-t} \quad \square$$

28. $\text{I.F.} = e^{\int \frac{2}{3}} = e^{\frac{2}{3}t}$ (3)

$$\left(e^{\frac{2}{3}t}y\right)' = e^{\frac{2}{3}t}\left(1-\frac{1}{2}t\right)$$

$$e^{\frac{2}{3}t}y = -\frac{3}{8}e^{\frac{2}{3}t}(2t-7) + C$$

$$y = -\frac{3}{8}(2t-7) + Ce^{-\frac{2}{3}t} \quad (*)$$

We want to find y_0 such that for some t_0,

$$\begin{cases} y(t_0) = 0 \\ y'(t_0) = 0 \end{cases}$$

Putting $t = t_0$ into the equation $y' + \frac{2}{3}y = 1 - \frac{t}{2}$

$$0 = 1 - \frac{t_0}{2}$$

$$t_0 = 2$$

Putting this into $(*)$,

$$0 = -\frac{3}{8}(2\cdot 2 - 7) + Ce^{-\frac{2\cdot 2}{3}}$$

$$0 = \frac{9}{8} + Ce^{-\frac{4}{3}}$$

$$C = -\frac{9}{8}e^{\frac{4}{3}}$$

Now we put this and $t=0$ into $(*)$, to

$$y_0 = y(0) = -\frac{3}{8}(-7) + \left(-\frac{9}{8}e^{\frac{4}{3}}\right)\cdot 1$$

$$= \frac{21}{8} - \frac{9}{8}e^{\frac{4}{3}}$$

$\square$

38. (a) $y' + p(t)y = 0$ (4)

$$\text{I.F.} = e^{\int p(t)dt} = \exp\left(\int p(t)dt\right)$$

$$\left(\exp\left(\int p(t)dt\right)y\right)' = 0$$

$$\exp\left(\int p(t)dt\right)y = A, \text{ a constant}$$

$$y = A\exp\left(-\int p(t)dt\right). \square$$

(b) Putting $y = A(t)\exp\left(-\int p(t)dt\right)$, into

$$y' + p(t)y = g(t),$$

or into $\left(\exp\left(\int p(t)dt\right)y\right)' = \left(\exp\left(\int p(t)dt\right)y\right)g(t)$

we have $A'(t) = g(t)\exp\left(\int p(t)dt\right)$ $\square$

(c) From (b),

$$A(t) = \int^t g(s)\exp\left(\int^s p(r)dr\right)ds.$$

so

$$y = \left(\int^t g(s)\exp\left(\int^s p(r)dr\right)ds\right)\exp\left(-\int^t p(s)ds\right)$$

$\square$

42 $y' + \frac{1}{2}y = \frac{3}{2}t^2$, so $\text{I.F.} = e^{\frac{1}{2}t}$ and

$$y = \left(\int \frac{3}{2}t^2 e^{\frac{1}{2}t}dt\right)e^{-\frac{1}{2}t}$$

$$= \left(3e^{\frac{1}{2}t}(t^2 - 4t + 8) + C\right)e^{-\frac{1}{2}t}$$ $\square$

2.2 3. $y' + y^2 \sin x = 0.$

For $y \neq 0$, $\dfrac{y'}{y^2} = -\sin x.$

Integrating, $-\dfrac{1}{y} = \cos x + C$

i.e. $y = -\dfrac{1}{\cos x + C}$ $\square$

6. $x y' = (1 - y^2)^{\frac{1}{2}}$

$$\frac{y'}{(1-y^2)^{\frac{1}{2}}} = \frac{1}{x}$$

Integrating,

$\sin^{-1}(y) = \log|x| + C$

i.e. $y = \sin\left(\log|x| + C\right)$ $\square$

7. $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^{y}}$

$\int (y + e^y)\,dy = \int (x - e^{-x})\,dx$

$\dfrac{y^2}{2} + e^y = \dfrac{x^2}{2} + e^{-x} + C$ $\square$

13. $$y' = \frac{2x}{y + x^2 y}$$

(6)

$$yy' = \frac{2x}{1+x^2}$$

Integrating,

$$\frac{y^2}{2} = \log(1+x^2) + C$$

As $y(0) = -2$,

$$\frac{(-2)^2}{2} = 0 + C$$

$$C = 2$$

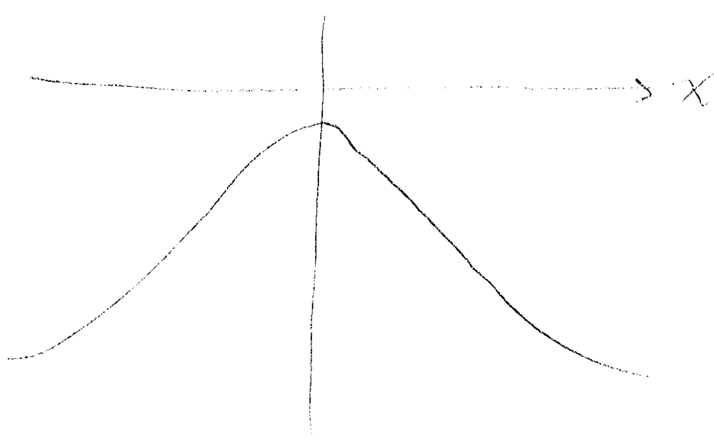
$$\therefore \quad y^2 = 2\log(1+x^2) + 4$$

$$y = \pm\sqrt{2\log(1+x^2) + 4}$$

But the initial condition $y(0) = -2$ forces us to take $-$ sign, hence

$$y = -\sqrt{2\log(1+x^2) + 4}$$

The solution is defined for all $x \in \mathbb{R}$.



19. $\sin 2x \, dx + \cos 3y \, dy = 0$ ⑦

Integrating,

$$-\frac{\cos 2x}{2} + \frac{\sin 3y}{3} + C = 0$$

Since $y(\frac{\pi}{2}) = \frac{\pi}{3}$,

$$-\frac{\cos \pi}{2} + \frac{\sin \pi}{3} + C = 0$$

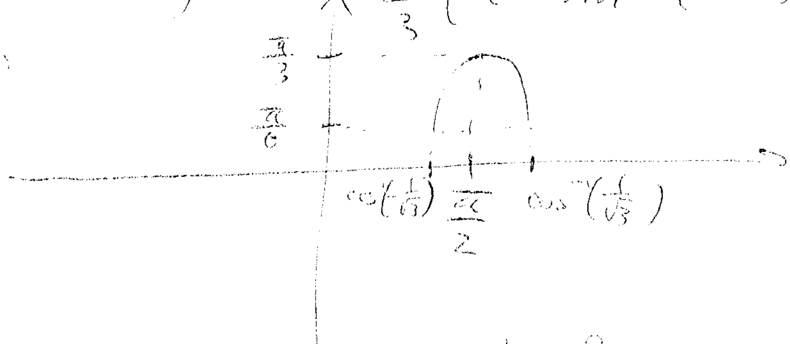
$$C = -\frac{1}{2}.$$

$$\therefore \quad \frac{\sin 3y}{3} = \frac{\cos 2x + 1}{2} = \cos^2 x$$

$$\pi - 3y = \cancel{\tfrac{1}{3}} \sin^{-1}(3\cos^2 x)$$

determined by initial condition

$$y = \frac{1}{3}\left(\pi - \sin^{-1}(3\cos^2 x)\right).$$



The solution is defined for x near $\frac{\pi}{2}$ that satisfies $3\cos^2 x \leq 1$

$$-\frac{1}{\sqrt{3}} \leq \cos x \leq \frac{1}{\sqrt{3}}$$

$$\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \leq x \leq \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

(note that cos is decreasing from 0 to π)

21. $$y' = \frac{1+3x^2}{3y^2-6y}, \quad y(0)=1.$$ (8)

Note that $3y^2-6y = 3y(y-2) = 0$ iff $y=0$ or 2. For $y \neq 0, 2$,

$$3y^2y' - 6yy' = 1+3x^2$$

Integrating

$$y^3 - 3y^2 = x + x^3 + C$$

Using $y(0)=1$,

$$1-3 = C$$

$$C = -2$$

$$\therefore \quad y^3 - 3y = x + x^3 - 2$$

$y=0$ iff $$x^3 + x - 2 = 0$$
$$(x-1)(x^2+x+2) = 0$$
$$x = 1$$

$y=2$ iff $$x^3 + x \overset{+2}{\cancel{-4}} = 0$$
$$(x+1)(x^2-x+2) = 0$$
$$\cancel{x_0 = x_0 = \sqrt[3]{\frac{18+\sqrt{327}}{9}} - \frac{1}{\sqrt[3]{3(18+\sqrt{327})}} \approx 1.3788}$$
$$x = -1$$

So the solution is defined for all $|x|<1$. □

33. $$\frac{dy}{dx} = \frac{4y-3x}{2x-y} = \frac{4\left(\frac{y}{x}\right)-3}{2-\frac{y}{x}}$$

Let $v = \frac{y}{x}$. Then $y = xv$ and
$y' = xv' + v$.

So, $$xv' + v = \frac{4v-3}{2-v}$$

$$xv' = \frac{4v-3-v(2-v)}{2-v}$$

$$= \frac{v^2+2v-3}{2-v}$$

$$\frac{2-v}{(v+3)(v-1)}v' = \frac{1}{x}.$$

Integrating,

$$\frac{1}{4}\left(\log|1-v| - 5\log|v+3|\right) = \log|x| + C.$$

$$\frac{1}{4}\log\left|\frac{1-\frac{y}{x}}{\left(\frac{y}{x}+3\right)^5}\right| = \log|x| + C.$$

or $$\left|1-\frac{y}{x}\right|^{\frac{1}{4}} = C'|x|\left|\frac{y}{x}+3\right|^{\frac{5}{4}} \quad (C'>0)$$

or $$\left|1-\frac{y}{x}\right| = C'|x|^4\left|\frac{y}{x}+3\right|^5$$

Summary of answers

1.3 4. order $=1$, nonlinear

6. order $=3$, linear

18. $r = 0, 1, 2$

2.1 16. $y = \frac{\sin t}{t^2}$

18. $y = \frac{1}{t^2}\left(-t\cos t + \sin t + \frac{\pi^2}{4} - 1\right)$

19. $y = -\frac{t+1}{t^4}e^{-t}$

28. $y_0 = \frac{3}{8}(7 - 3e^{\frac{4}{3}}) = \frac{21}{8} - \frac{9}{8}e^{\frac{4}{3}}$ (≈ -1.64)

38. $y = e^{-\int^t p} \int^t g(s) e^{\int^s p}\, ds$

42. $y = 3(t^2 - 4t + 8) + Ce^{-\frac{t}{2}}$

2.2 3. $y = -\frac{1}{\cos x + C}$ ($y \equiv 0$ corresponds to $C = \infty$)

6. $y = \sin(\log|x| + C)$

7. $\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + C$

13. $y = -\sqrt{2\log(1+x^2) + 4}$, $x \in \mathbb{R}$

19. $y = \frac{1}{3}\left(\pi - \sin^{-1}(3\cos^2 x)\right)$, $x \in \left(\cos^{-1}\frac{1}{\sqrt{3}}, \cos^{-1}\frac{-1}{\sqrt{3}}\right)$

21. $y^3 - 3y = x^3 + x - 2$, $|x| < 1$

33. $\frac{1}{4}\left(\log\left|1 - \frac{y}{x}\right| - 5\left|\frac{y}{x} + 3\right|\right) = \log|x| + C$

or $\left|1 - \frac{y}{x}\right| = C|x|^4\left|\frac{y}{x} + 3\right|^5$

or $|x - y| = C|y + 3x|^5$