MATH 256:103 Solution 2

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2.3 8(a) $S' = rS + k$, $S(0)=0$

$(e^{-rt}S)' = ke^{-rt}$

$e^{-rt}S = k\left[\frac{e^{-rt}}{-r}\right]_0^t = \frac{k}{r}(1-e^{-rt})$

$\boxed{S = \frac{k}{r}(e^{rt}-1)}$

(b) $10^6 = \frac{k}{0.075}(e^{0.075(40)}-1)$

$\boxed{k = \frac{75000}{e^3-1}}$

(c) $10^6 = \frac{2000}{r}(e^{r(40)}-1)$

$\boxed{e^{40r} - 500r - 1 = 0}$

Software $\Rightarrow$ $\boxed{r \approx 9.77\%}$

20.(a) $mv' = -\frac{1}{30}v - mg$

$v' + \frac{1}{30m}v = -g$, i.e. $v' + \frac{2}{9}v = -g$

$(e^{\frac{2}{9}t}v)' = -ge^{\frac{2}{9}t}$

$e^{\frac{2}{9}t}v - 20 = -\frac{2g}{5}(e^{\frac{2}{9}t}-1)$

$v = 20e^{-\frac{2}{9}t} - \frac{2g}{5}(1-e^{-\frac{2}{9}t})$

$= \frac{-2g}{5} + (20+\frac{2g}{5})e^{-\frac{2}{9}t}$

$v(t_1)=0 \Rightarrow t_1 \approx 1.683$

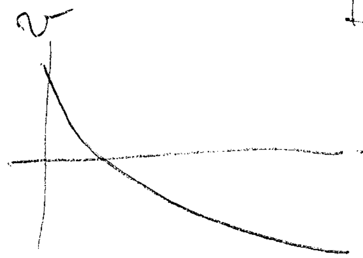
$x = -\frac{2g}{5}t + 30 - \frac{9}{2}(20+\frac{2g}{5})(e^{-\frac{2}{9}t}-1)$

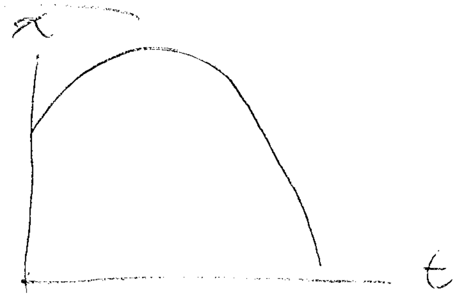
$$x = 30 - \frac{29}{5}t + 9\left(10 + \frac{9}{5}\right)\left(1 - e^{-\frac{2}{9}t}\right)$$

$$\boxed{x(t_1) \approx 45.78}$$

(b). $x(t_2) = 0 \Longrightarrow \boxed{t_2 \approx 5.128}$

(c)





2.4 28. $v = y^{-2}$, $\frac{dv}{dt} = -\frac{2}{y^3}\frac{dy}{dt}$, $\frac{dy}{dt} = -\frac{y^3}{2}\frac{dv}{dt}$.

$$t^2\left(-\frac{v^{-3/2}}{2}\frac{dv}{dt}\right) + 2t\left(v^{-\frac{1}{2}}\right) - v^{-\frac{3}{2}} = 0$$

$$-\frac{1}{2}t^2\frac{dv}{dt} + 2tv - 1 = 0$$

$$\frac{dv}{dt} - \frac{4}{t}v + \frac{2}{t^2} = 0$$

$$\left(t^{-4}v\right)' = \frac{-2}{t^6}$$

$$t^{-4}v = \frac{2}{5}t^{-5} + C \qquad t^{-4}v = +\frac{10}{t^5} + C$$

$$= \frac{2\cancel{t} + 5Ct^5}{5t^5} \qquad v = +\frac{10}{t} + \cancel{}Ct^4$$

$$\frac{1}{y^2} = \frac{2 + 5Ct^5}{5t} \qquad \frac{1}{y^2} = \frac{10 + Ct^5}{t}$$

$$\boxed{y = \pm\sqrt{\frac{5t}{2 + Ct^5}}}$$

2.4 30. $v = y^{-2}, \quad \frac{dy}{dt} = -\frac{v^{-\frac{3}{2}}}{2}\frac{dv}{dt}$.

Sub. $\Rightarrow \quad v' + 2\varepsilon v = 2\sigma$.

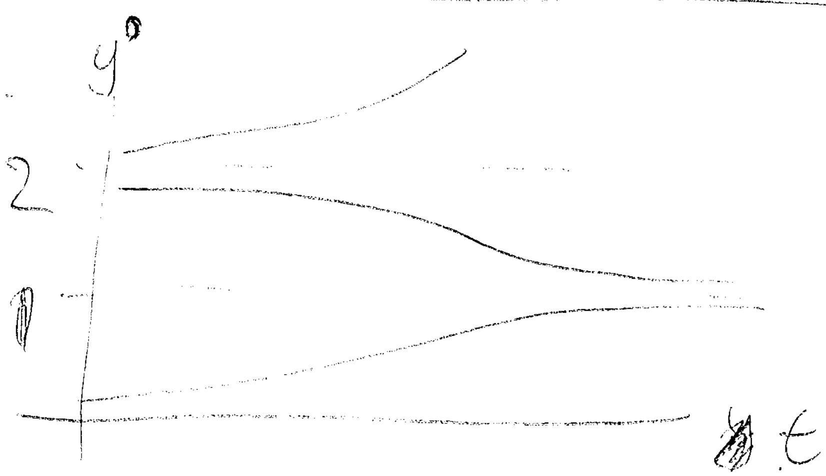
$$(e^{2\varepsilon t} v)' = 2\sigma e^{2\varepsilon t}$$

$$e^{2\varepsilon t} v = \frac{\sigma}{\varepsilon} e^{2\varepsilon t} + C.$$

$$\frac{1}{y^2} = v = \frac{\sigma}{\varepsilon} + Ce^{-2\varepsilon t}$$

$$\boxed{y = \pm\sqrt{\frac{\varepsilon}{\sigma + C\varepsilon e^{-2\varepsilon t}}}}$$

2.5 3.



$y=1$ asym. stable

$y=0$, $y=2$ unstable.

2.5 15.(a) By ~~数~~ Eq. (13) with $y_0 = K/3$,

$$t = -\frac{1}{r}\log\left|\frac{\frac{1}{3}(1-\frac{y}{K})}{\frac{y}{K}(1-\frac{1}{3})}\right|$$

$y = 2y_0$:

$$\tau = -\frac{1}{r}\log\left|\frac{\frac{1}{3}(1-\frac{2}{3})}{\frac{2}{3}(1-\frac{1}{3})}\right| = \frac{\log 4}{r}.$$

If $r = 0.025$, $\boxed{\tau \approx 55.45}$

(b). (13) with $y_0/K = \alpha$, $y/K = \beta$:

$$T = -\frac{1}{r}\log\left|\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right|$$

$$\to \infty \quad \text{as} \quad \alpha \to 0 \quad \text{or} \quad \beta \to 1.$$

Given α, β, r, $\boxed{\tau \approx 125.78 \text{ yr}}$

17.(a) $u = \log\frac{y}{K}$, $u' = \frac{y'}{y}$.

$$u' = -ru.$$

$$u = u_0 e^{-rt}$$

$$\log\frac{y}{K} = \log\frac{y_0}{K}e^{-rt}$$

$$\boxed{y = K\exp\left(\log\frac{y_0}{K}e^{-rt}\right)}.$$

(b). $\boxed{y(2) \approx 57.58 \cdot 10^6}$.

(c). $$t = -\frac{1}{r}\log\frac{\log\frac{y}{K}}{\log\frac{y_0}{K}}, \quad y(\tau) = 0.75K$$

$\Rightarrow \boxed{\tau \approx 2.21 \text{ years}}$

3.1 3. char. eq. $6\lambda^2-\lambda-1=0 \Rightarrow \lambda=-\frac{1}{3}, \frac{1}{2}$.

$$\boxed{y=c_1 e^{-\frac{1}{3}t}+c_2 e^{\frac{1}{2}t}}$$

5. char. eq. $\lambda^2+5\lambda=0 \Rightarrow \lambda=0,-5$

$$\boxed{y=c_1+c_2 e^{-5t}}$$

14. $2\lambda^2+\lambda-4=0 \Rightarrow \lambda_\pm=\frac{-1\pm\sqrt{33}}{4}t$.

$$y=c_1 e^{\lambda_+ t}+c_2{}^{\lambda_- t}$$

$y(0)=0,\ y'(0)=1$

$$\Rightarrow \boxed{y=\frac{2}{\sqrt{33}}\left(e^{\lambda_+ t}-e^{\lambda_- t}\right)}$$

$\lambda_+>0 \Rightarrow \boxed{y\to\infty \text{ as } t\to\infty}$

15. $\lambda=-9, 1$.

$$y=c_1 e^{-9t}+c_2 e^{t}$$

$y(\cancel{t}1)=1,\ y'(1)=0$

$$\Rightarrow \boxed{y=\frac{e^9}{10}e^{-9t}+\frac{9}{10e}e^{t}}$$

$$\boxed{y\to\infty \text{ as } t\to\infty}$$

21. $\lambda=2,-1$

$y=c_1 e^{2t}+c_2 e^{-t},\quad y'=2c_1 e^{2t}-c_2 e^{-t}$.

$y(0)=\alpha,\ y'(0)=-2$

$$\Rightarrow y=\frac{\alpha+2}{2}e^{2t}+\alpha e^{-t}$$

Need $\boxed{\alpha=-2}$ s.t. $y\to 0$ as $t\to\infty$

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26. (a) $y = c_1 e^{\cancel{-t}-2t} + c_2 e^{-3t}$

$\cancel{\Xi}$, $y(0) = 2$, $y'(0) = \beta$

$\Longrightarrow \boxed{y = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}}$

(b) $y'(t_m) = 0 \Longrightarrow \boxed{t_m = \log \frac{12+3\beta}{12+2\beta}}$

$\boxed{y_m = y(t_m) = \frac{4(6+\beta)^3}{27(4+\beta)^2}}$.

(c) $y_m = \frac{4(6+\beta)^3}{27(4+\beta)^2} \geq 4 \Longrightarrow \boxed{\beta \geq 6+6\sqrt{3}}$

(d) $\boxed{\lim_{\beta\to\infty} t_m = \log \frac{3}{2}}$, $\boxed{\lim_{\beta\to\infty} y_m = \infty}$.

3.2 5. $W = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t(\sin t + \cos t) & e^t(\cos t - \sin t) \end{vmatrix} = \boxed{-e^{2t}}$

7. $3e^{4t} = W = \begin{vmatrix} \cancel{3e^{4t}}\, e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = e^{2t}(g' - 2g)$

$g' - 2g = 3e^{2t}$

Solving, $\boxed{g = 3te^{2t} + Ce^{2t}}$

22. $y = c_1 e^{-2t} + c_2 e^{+t}$

$W(e^{-2t}, e^{+t}) \neq 0 \Rightarrow \{e^{-2t}, e^{+t}\}$ is F.S.S.

Solving $\begin{cases} y_1(0) = 1 \\ y_1'(0) = 0 \end{cases}$ and $\begin{cases} y_2(0) = 0 \\ y_2'(0) = 1 \end{cases}$,

$\boxed{y_1 = \frac{1}{3}e^{-2t} + \frac{2}{3}e^t, \quad y_2 = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t}$

24. $W(y_1, y_2) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2 \neq 0$, $\boxed{\text{yes}}$

29. $W = c e^{-\int \frac{t+2}{t} dt} = \boxed{c t^2 e^{t}}$

31. $W = c e^{-\int \frac{1}{x} dt} = \boxed{\frac{c}{x}}$

33. $\underline{p y'' + p' y' + q y = 0}$

$$\boxed{W = c e^{-\int \frac{p'}{p}} = \frac{c}{p(t)}}$$

35. $W(t) = c e^{-\int \frac{-2}{t^2}} = c e^{-\frac{2}{t}}$

$W(2) = 3 \Rightarrow c = 3e.$

$W(4) = \cancel{3ee} \; 3e^{1-\frac{1}{2}} = \boxed{3\sqrt{e}}.$