olutions to Assignment 3

characteristic equation

$$r^2+2r+3=0 \quad , \quad r = \frac{-2 \pm \sqrt{4-12}}{2}$$

So $r = -1 \pm \sqrt{2}\, i$

$\lambda = -1, \quad \mu = \sqrt{2}$

$$y_1 = e^{\lambda t}\cos\mu t = e^{-t}\cos(\sqrt{2}t), \quad y_2 = e^{-t}\sin(\sqrt{2}t)$$

$$y = c_1 e^{-t}\cos(\sqrt{2}t) + c_2 e^{-t}\sin(\sqrt{2}t)$$

$$y(0) = -1 \implies c_1 = -1$$

$$y'(0) = 2 \implies -c_1 \sqrt{2}\, c_2 = 2 \implies c_2 = \frac{1}{\sqrt{2}}$$

So

$$y = -e^{-t}\cos(\sqrt{2}t) + \frac{1}{\sqrt{2}} e^{-t}\sin\sqrt{2}t$$

(a) This is Euler's type

characteristic equation

$$r(r-1)+3r-2=0$$

$$r^2+2r-2=0 \implies r = -1 \pm \sqrt{3}, \quad \cancel{\lambda = -1, \mu = \sqrt{3}}$$

$$\cancel{y_1 = t^{-1}\cos(\sqrt{3}\ln t), \quad y_2 = t^{-1}\sin(\sqrt{3}\ln t)}$$

$$\cancel{y = c_1 t^{-1}\cos(\sqrt{3}\ln t) + c_2 t^{-1}\sin(\sqrt{3}\ln t)}$$

$$y = c_1 t^{-1-\sqrt{3}} + c_2 t^{-1+\sqrt{3}}$$

(b) Euler's Type

Characteristic Equation

$$2r(r-1)-4r+1 = 0$$

$$2r^2-6r+1=0 \qquad r = \frac{6 \pm \sqrt{36-8}}{4} = \frac{3}{2} \pm \frac{\sqrt{7}}{2}$$

$$r_1 = \frac{3}{2} + \frac{\sqrt{7}}{2} \qquad r_2 = \frac{3}{2} - \frac{\sqrt{7}}{2}$$

$$y_1 = t^{r_1} \qquad y_2 = t^{r_2}$$

$$\boxed{y = c_1 t^{\frac{3}{2}+\frac{\sqrt{7}}{2}} + c_2 t^{\frac{3}{2}-\frac{\sqrt{7}}{2}}}$$

Characteristic equation:

$$r^2-6r+9=0 \Rightarrow r_1=r_2=3$$

$$y_1=e^{3t}, \quad y_2 = t\,e^{3t}$$

$$y = c_1 e^{3t} + c_2 t e^{3t}$$

$$y(0)=0 \Rightarrow c_1 = 0$$

$$y'(0)=1 \Rightarrow c_2 = 1 \Rightarrow c_2 = 1$$

$$\boxed{y = t e^{3t}}$$

$y_2 = v\, y_1$, and $v$ satisfies

$$v'' + \left(\frac{2y_1'}{y_1} + p\right) v' = 0$$

(a) $p = -\frac{1}{t}, \quad y_1 = \sin t^2$

So $$v'' + \left( 2 \cdot \frac{2t \cos t^2}{\sin t^2} - \frac{1}{t} \right) v' = 0$$

$$v' = e^{-\int \left( \frac{4t \cos t^2}{\sin t^2} - \frac{1}{t} \right) dt}$$

$$= \frac{t}{(\sin t^2)^2}$$

$$v = \int \frac{t}{(\sin t^2)^2} dt \overset{u = t^2}{=\!=\!=} \frac{1}{2} \int \frac{du}{\sin^2 u}$$

$$= -\frac{1}{2} \cot t^2$$

Thus $\boxed{y_2 = -\frac{1}{2} \cot t^2 \ \sin t^2 = -\frac{1}{2} \cos t^2}$

(b) $p = -\frac{t}{t-1}, \quad y_1 = e^t$

$$v'' + \left( \frac{2 e^t}{e^t} - \frac{t}{t-1} \right) v' = 0$$

$$v' = e^{-\int \left(2 - \frac{t}{t-1}\right) dt} = e^{-t} |t-1|$$

$$v = \int e^{-t} (t-1)\, dt = -t e^{-t}.$$

$\boxed{y_2 = v y_1 = -t}$

5. By definition

$$W = y_1 y_2' - y_2 y_1'$$

$$= y_1 (v y_1)' - y_2 y_1'$$

$$= y_1 (v' y_1 + v y_1') - \cancel{y_2} v y_1 y_1'$$

$$= v' y_1^2 + v y_1 y_1' - v y_1 y_2'$$

$$= v' y_1^2$$

So $v' = \dfrac{W}{y_1^2}$

6. Use Problem 5

$$v' = \frac{W}{y_1^2}$$

By Abel's formula $\quad W = e^{-\int p}$

where $\quad p = \dfrac{t}{t^2} = \dfrac{1}{t}$

so $\quad W = e^{-\int \frac{1}{t} dt} = \dfrac{1}{t}$

$$v' = \frac{\frac{1}{t}}{(t^{-\frac{1}{2}} \sin t)^2} = \frac{1}{(\sin t)^2}$$

so $\quad v = \int \dfrac{1}{\sin^2 t} = -\cot t$

$$y_2 = v y_1 = -\cot t \; t^{-\frac{1}{2}} \sin t = -t^{-\frac{1}{2}} \cos t$$