3.7

?/8. 1

26. (a) $mu'' + \gamma u' + ku = 0$, $u(0) = u_0$, $u'(0) = v_0$, $\gamma^2 < 4km$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{4km-\gamma^2}\, i}{2m}$$

$$u = c_1 e^{-\frac{\gamma}{2m}t} \cos \frac{\sqrt{4km-\gamma^2}}{2m} t + c_2 e^{-\frac{\gamma}{2m}t} \sin \frac{\sqrt{4km-\gamma^2}}{2m} t$$

$u_0 = u(0) = c_1$

$$v_0 = u'(0) = c_1\left(-\frac{\gamma}{2m}\right) + c_2\left(\frac{\sqrt{4km-\gamma^2}}{2m}\right) \Rightarrow c_2 = \frac{2mv_0 + \gamma u_0}{\sqrt{4km-\gamma^2}}.$$

$$\boxed{u = u_0 e^{-\frac{\gamma}{2m}t} \cos \frac{\sqrt{4km-\gamma^2}}{2m} t + \frac{2mv_0 + \gamma u_0}{\sqrt{4km-\gamma^2}} e^{-\frac{\gamma}{2m}t} \sin \frac{\sqrt{4km-\gamma^2}}{2m} t.}$$

(b) $u = Re^{-\frac{\gamma}{2m}t} \cos\left(\frac{\sqrt{4km-\gamma^2}}{2m} t - \delta\right)$

$$\Rightarrow \begin{cases} R\cos\delta = u_0 \\ R\sin\delta = \dfrac{2mv_0 + \gamma u_0}{\sqrt{4km-\gamma^2}} \end{cases}$$

$$\Rightarrow \begin{cases} R = \sqrt{u_0^2 + \dfrac{(2mv_0 + \gamma u_0)^2}{4km-\gamma^2}} = \sqrt{\dfrac{4m^2v_0^2 + 4m\gamma u_0 v_0 + 4km u_0^2}{4km-\gamma^2}} \\ \delta \text{ determined by above (modulo } 2\pi) \end{cases}$$

(c) $R \to \infty$ if $\gamma \to \sqrt{4km}$, $R \to 0$ if $\gamma \to \infty$

2

3.8 17.

$$u'' + \frac{1}{4}u' + 2u = 2\cos(\omega t), \quad u(0)=0, \quad u'(0)=2$$

(a) $4r^2 + r + 8 = 0 \Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{127}\, i}{8}$

$$u_h = C_1 e^{-\frac{1}{8}t}\cos\frac{\sqrt{127}}{8}t + C_2 e^{-\frac{1}{8}t}\sin\frac{\sqrt{127}}{8}t$$

Let $u_p = A\cos\omega t + B\sin\omega t$

$$u_p'' + \frac{1}{4}u_p' + 2u_p = \cos\omega t\left(-A\omega^2 + \frac{1}{4}B\omega + 2A\right) + \sin\omega t\left(-B\omega^2 - \frac{1}{4}A\omega + 2B\right)$$

$$\Rightarrow \begin{cases} (-\omega^2+2)A + \frac{1}{4}\omega B = 2 \\ -\frac{1}{4}\omega A + (-\omega^2+2)B = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{(-\omega^2+2)^2 + (\frac{1}{4}\omega)^2}\begin{pmatrix} -\omega^2+2 & -\frac{1}{4}\omega \\ \frac{1}{4}\omega & -\omega^2+2 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow A = \frac{2(-\omega^2+2)}{(-\omega^2+2)^2+(\frac{1}{4}\omega)^2}, \quad B = \frac{-\frac{1}{2}\omega}{(-\omega^2+2)^2+(\frac{1}{4}\omega)^2}$$

$$u = C_1 e^{-\frac{1}{8}t}\cos\frac{\sqrt{31}}{8}t + C_2 e^{-\frac{1}{8}t}\sin\frac{\sqrt{31}}{8}t + \frac{2(-\omega^2+2)\cos\omega t - \frac{1}{2}\omega\sin\omega t}{(-\omega^2+2)^2 + (\frac{1}{4}\omega)^2}$$

as $t \to \infty$, $\boxed{u \sim \frac{2(-\omega^2+2)\cos\omega t - \frac{1}{2}\omega\sin\omega t}{(-\omega^2+2)^2+(\frac{1}{4}\omega)^2}} =: u_s$

(b) $u_s = \frac{2}{\sqrt{(-\omega^2+2)^2+(\frac{1}{4}\omega)^2}}\cos(\omega t - \delta)$

where $\begin{cases} \cos\delta = \frac{-\omega^2+2}{\sqrt{(-\omega^2+2)^2+(\frac{1}{4}\omega)^2}} \\ \sin\delta = \frac{-\frac{1}{4}\omega}{\sqrt{(-\omega^2+2)^2+(\frac{1}{4}\omega)^2}} \end{cases}$

$\boxed{\text{Amplitude } A = \frac{2}{\sqrt{(-\omega^2+2)^2+(\frac{1}{4}\omega)^2}}}$

(d) A is maximized when

$$(-\omega^2+2)^2+\left(\tfrac{1}{4}\omega\right)^2 = \omega^4 - \tfrac{15}{4}\omega^2 + 4$$

is minimized, i.e. when $\omega^2 = \frac{15}{8}$, $\boxed{\omega = \sqrt{\frac{15}{8}}}$

$$\boxed{\max A} = A\left(\sqrt{\tfrac{15}{8}}\right) = \frac{2}{\sqrt{4-\left(\frac{15}{8}\right)^2}} = \boxed{\frac{16}{\sqrt{31}}}$$

7.3

15. $x^{(1)} = e^t \begin{pmatrix} 1 \\ t \end{pmatrix} = e^t x^2$ at each t.

If $x^{(1)}(t)$ and $x^{(2)}(t)$ are lin. indep,

$$c_1 x^1 + c_2 x^2 = 0 \quad \text{on } 0 \le t \le 1.$$

i.e. $(c_1 e^t + c_2)\begin{pmatrix} 1 \\ t \end{pmatrix} = 0$ on $0 \le t \le 1$.

which is impossible.

16. $A - \lambda I = \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix}$

$\det(A-\lambda I) = 0 \Rightarrow (\lambda^2 - 6\lambda + 5) + 3 = 0 \Rightarrow \lambda = 2, 4.$

$\boxed{\lambda=2}$ $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} x_2 = 0 \Rightarrow \boxed{x_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$

$\boxed{\lambda=4}$ $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} x_4 = 0 \Rightarrow \boxed{x_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

17. $A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}$

$\det(A-\lambda I) = 0 \Rightarrow \boxed{\lambda_\pm = 1 \pm 2i}$

$(A-\lambda_\pm I)x_\pm = 0 \Rightarrow \boxed{x_+ = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}}, \boxed{x_- = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}}$

20. $\begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & 1-\lambda \end{pmatrix}$.

$\det(A-\lambda I)=0 \Rightarrow \boxed{\lambda_\pm = \pm 2.}$

$(A-\lambda_\pm I)x_\pm = 0 \Rightarrow \boxed{x_+ = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix},\quad x_- = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}.}$ □.

22. $\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix}$.

$\det(A-\lambda I)=0 \Rightarrow \boxed{\lambda = 1, 2, 3.}$

$\Rightarrow \boxed{x_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix},\quad x_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix},\quad x_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.}$ □.

25. $\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix}$

$\det(A-\lambda I)=0 \Rightarrow \boxed{\lambda = -1, -1, 8.}$

$\Rightarrow \boxed{x_{-1}^{(1)} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix},\quad x_{-1}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\quad x_8 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.}$ □.

7.4

2. (a). $W = x_1^{(1)} x_2^{(2)} - x_1^{(2)} x_2^{(1)}$

$$\frac{dW}{dt} = \frac{dx_1^{(1)}}{dt} x_2^{(2)} - \frac{dx_1^{(2)}}{dt} x_2^{(1)} + x_1^{(1)} \frac{dx_2^{(2)}}{dt} - x_1^{(2)} x_2^{(1)}$$

$$= \begin{vmatrix} \frac{dx_1^{(1)}}{dt} & \frac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \frac{dx_2^{(1)}}{dt} & \frac{dx_2^{(2)}}{dt} \end{vmatrix}.$$

(5)

(b). $\frac{dW}{dt} = $ (RHS in (a)).

$$= \begin{vmatrix} p_{11}x_1^{(1)} + p_{12}x_2^{(1)} & p_{11}x_1^{(2)} + p_{12}x_2^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}$$

$$+ \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ p_{21}x_1^{(1)} + p_{22}x_2^{(1)} & p_{21}x_1^{(2)} + p_{22}x_2^{(2)} \end{vmatrix}.$$

= (terms with p_{12} are gone by prop. of det).

$= p_{11}W + p_{22}W$.

(c) $W = ce^{\int (p_{11}+p_{22})\,dt}$.

(d) Similar. Get $W = ce^{\int \mathrm{tr}P\,dt}$. □.

3. Follows from 2(b).

4. (i) $\Rightarrow$ ~~$W = c_1 e^{\int p}$~~ $W[y^{(1)}, y^{(2)}] = c_1 e^{-\int p}$.

(ii) $\Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$\overset{Q2(b)}{\Longrightarrow} W[x^{(1)}, x^{(2)}] = c_2 e^{-\int p}$. □.

7.5

1. ~~$\lambda_1 = -1,$~~ $\lambda = -1, 2,\ x_{-1} = \begin{pmatrix}1\\2\end{pmatrix},\ x_2 = \begin{pmatrix}2\\1\end{pmatrix}$

$$\boxed{\begin{array}{l} x = c_1\begin{pmatrix}1\\2\end{pmatrix}e^{-t} + c_2\begin{pmatrix}2\\1\end{pmatrix}e^{2t} \\ |x| \xrightarrow[t\to\infty]{} \infty, \text{ asymptotic to } y = 2x. \end{array}}$$

if $c_2 \neq 0$.
o/w $x \to 0$.

$\square$.

3. $\lambda = \pm 1,\ x_1 = \begin{pmatrix}1\\1\end{pmatrix},\ x_{-1} = \begin{pmatrix}1\\3\end{pmatrix}.$

$$\boxed{\begin{array}{l} x = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t} \\ |x| \xrightarrow[t\to\infty]{} \infty, \text{ asym. to } y = x \end{array}}$$

if $c_1 \neq 0$
o/w $x \to 0$

$\square$

5. $\lambda = -3, -1,\ x_{-3} = \begin{pmatrix}1\\-1\end{pmatrix},\ x_{-1} = \begin{pmatrix}1\\1\end{pmatrix}.$

$$\boxed{\begin{array}{l} x = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{-t} \\ x \xrightarrow[t\to\infty]{} 0. \end{array}}$$

$\square$

11. $\lambda = 4, 1, -1.$

$x_4 = \begin{pmatrix}1\\1\\1\end{pmatrix},\ x_1 = \begin{pmatrix}1\\-2\\1\end{pmatrix},\ x_{-1} = \begin{pmatrix}1\\0\\-1\end{pmatrix}.$

$$\boxed{x = c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{4t} + c_2\begin{pmatrix}1\\-2\\1\end{pmatrix}e^{t} + c_3\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-t}}$$

$\square$

15. $\lambda = 2, 4$, $x_2 = \begin{pmatrix}1\\3\end{pmatrix}$, $x_4 = \begin{pmatrix}1\\1\end{pmatrix}$.

$$x = c_1\begin{pmatrix}1\\3\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$$

$$\begin{pmatrix}2\\-1\end{pmatrix} = x(0) = c_1\begin{pmatrix}1\\3\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1 & 1\\3 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = \frac{1}{-2}\begin{pmatrix}1 & -1\\-3 & 1\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}-\frac{3}{2}\\\frac{7}{2}\end{pmatrix}.$$

$$\boxed{x = -\frac{3}{2}\begin{pmatrix}1\\3\end{pmatrix}e^{2t} + \frac{7}{2}\begin{pmatrix}1\\1\end{pmatrix}e^{4t}}$$

$\square$

17. $\lambda = 1, 2, 3$

$$x_1 = \begin{pmatrix}0\\-2\\1\end{pmatrix},\quad x_2 = \begin{pmatrix}1\\1\\0\end{pmatrix},\quad x_3 = \begin{pmatrix}0\\2\\-1\end{pmatrix}$$

$$x = c_1\begin{pmatrix}0\\-2\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t} + c_3\begin{pmatrix}0\\2\\-1\end{pmatrix}e^{3t}$$

$$x(0) = \begin{pmatrix}2\\0\\1\end{pmatrix} \Rightarrow \boxed{x = \cancel{\begin{pmatrix}0\\-2\\1\end{pmatrix} + 2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t}}\ \begin{pmatrix}0\\-2\\1\end{pmatrix}e^{t} + 2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t}}$$

$\square$

19. $tx' = Ax$

$$x = \xi t^r \Rightarrow t(\xi r t^{r-1}) = A\xi t^r$$

$$\Rightarrow (A - rI)\xi = 0.$$

$\square$

20. $A - rI = \begin{pmatrix}2-r & -1\\3 & -2-r\end{pmatrix}$, $\det(A - rI) = 0 \Rightarrow r = 1, -1$.

$$x_1 = \begin{pmatrix}1\\1\end{pmatrix},\quad x_2 = \begin{pmatrix}1\\3\end{pmatrix} \Rightarrow \boxed{x = c_1\begin{pmatrix}1\\1\end{pmatrix}t + c_2\begin{pmatrix}1\\3\end{pmatrix}t^{-1}}$$

$\square$