or $x = c_1\begin{pmatrix}2\cos t - \sin t\\ \cos t\end{pmatrix} + c_2\begin{pmatrix}2\sin t + \cos t\\ \sin t\end{pmatrix}$.

3(a). Similar.

$$\boxed{x = c_1\begin{pmatrix}5\cos t\\ 2\cos t + \sin t\end{pmatrix} + c_2\begin{pmatrix}5\sin t\\ -\cos t + 2\sin t\end{pmatrix}}$$

5.(a). $x = c_1 e^{-t}\begin{pmatrix}\cos t\\ 2\cos t + \sin t\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}\sin t\\ -\cos t + 2\sin t\end{pmatrix}$.

7. $x = c_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + c_2 e^t\begin{pmatrix}0\\ \cos 2t\\ \sin 2t\end{pmatrix} + c_3 e^t\begin{pmatrix}0\\ \sin 2t\\ -\cos 2t\end{pmatrix}$.

13.(a). $\det\begin{pmatrix}\alpha-\lambda & 1\\ -1 & \alpha-\lambda\end{pmatrix} = 0$

$\Rightarrow (\lambda-\alpha)^2 + 1 = 0$

$\Rightarrow \boxed{\lambda = \alpha \pm i}$

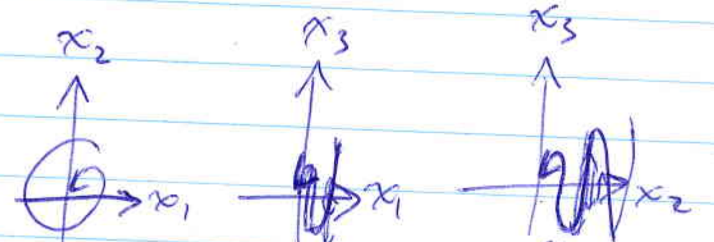
(b). $\alpha = 0$

$\left(\begin{array}{lll}\alpha > 0 & \Rightarrow: |x| \to \infty & \text{along a spiral}\\ \alpha = 0 & : \text{center} & \\ \alpha < 0 & : |x| \to 0. & \text{along a spiral}\end{array}\right)$.

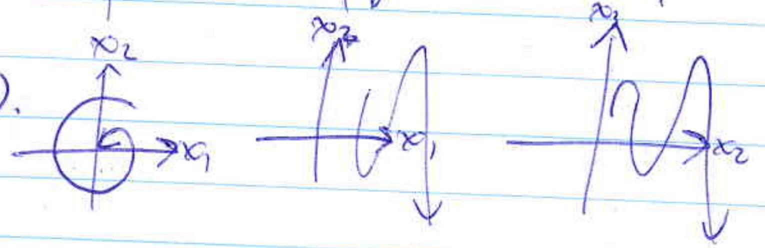
17.(a). $r = -1 \pm \sqrt{-\alpha}$

(b) $\alpha = -1, 0$.

23.(a) $r = -\frac{1}{4}, -\frac{1}{4} \pm i$. (b).



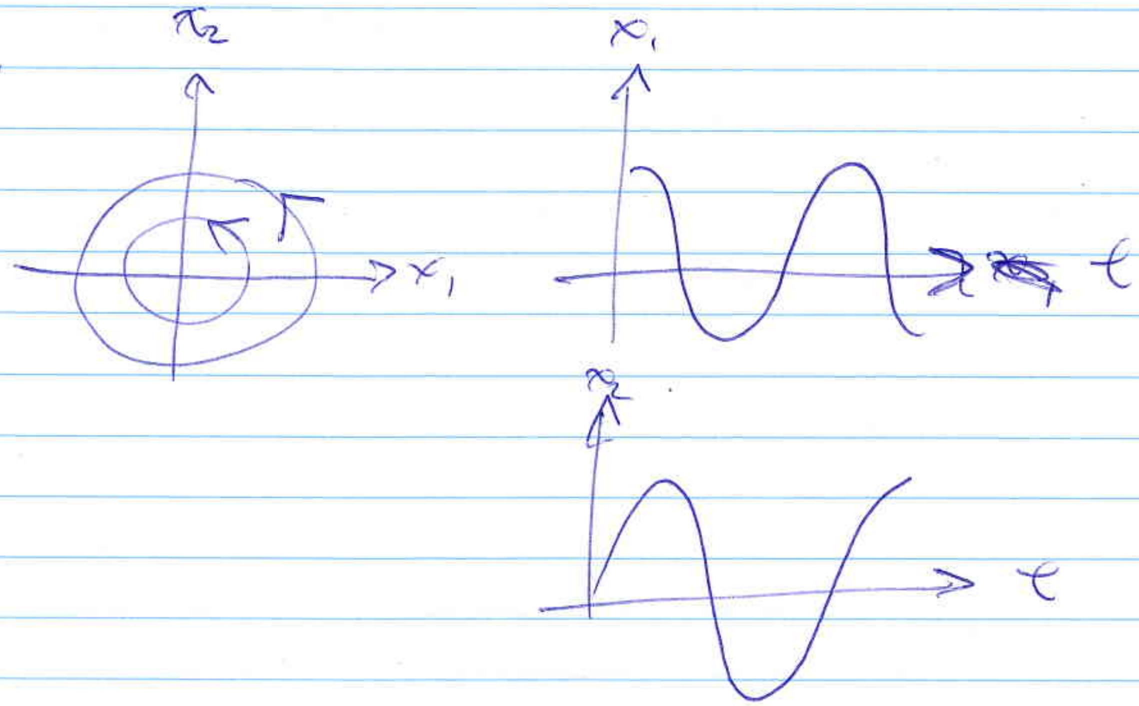
24(a). $r = \frac{1}{10}, -\frac{1}{4} \pm i$. (b).



28. (a). $\begin{cases} m x_2' + kx_1 = 0\\ x_2 = x_1'\end{cases} \Rightarrow x' = \begin{pmatrix}0 & 1\\ -\frac{k}{m} & 0\end{pmatrix}x$.

(b). $\det\begin{pmatrix}-\lambda & 1\\ -\frac{k}{m} & -\lambda\end{pmatrix} = 0 \Rightarrow \lambda = \pm i\sqrt{\frac{k}{m}}$.

28. (c).



(d). $\sqrt{\frac{k}{m}}$ is the natural frequency.

7.7

1 $\det\begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = -1, 2.$

$\lambda_1 = -1: \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} x_1 = 0 \Rightarrow x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = 2: \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} x_2 = 0 \Rightarrow x_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

(a) a fundamental matrix is

$$\Psi(t) = \begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix}.$$

(b). the required fundamental matrix is

$$\Phi(t) = \Psi(t)\Psi^{-1}(0) = \begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix} \frac{1}{-3} \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

$$= \frac{1}{3} \begin{pmatrix} -e^{-t} + 4e^{2t} & 2e^{-t} - 2e^{2t} \\ -2e^{-t} + 2e^{2t} & 4e^{-t} - e^{2t} \end{pmatrix}.$$

$\square$

3. Similar.

$$\boxed{\Phi(t) = \frac{1}{2}\begin{pmatrix} 3e^{t}-e^{-t} & -e^{t}+e^{-t} \\ 3e^{t}-3e^{-t} & -e^{t}+3e^{-t} \end{pmatrix}}$$

5. $$\boxed{\Phi(t) = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}}$$

7. $$\Phi(t) = \frac{1}{2}\begin{pmatrix} -e^{2t}+3e^{4t} & e^{2t}-e^{4t} \\ -3e^{2t}+3e^{4t} & 3e^{2t}-e^{4t} \end{pmatrix}$$

9. $$\boxed{\Phi(t) = \begin{pmatrix} -2e^{-2t}+3e^{-t} & -e^{-2t}+e^{-t} & -e^{-2t}+e^{-t} \\ \frac{5}{2}e^{2t}-4e^{-t}+\frac{3}{2}e^{2t} & \frac{5}{4}e^{-2t}-\frac{4}{3}e^{-t}+\frac{13}{12}e^{2t} & \frac{5}{4}e^{-2t}-\frac{4}{3}e^{-t}+\frac{1}{12}e^{2t} \\ \frac{7}{2}e^{-2t}-2e^{-t}-\frac{3}{2}e^{2t} & \frac{7}{4}e^{-2t}-\frac{2}{3}e^{-t}-\frac{13}{12}e^{2t} & \frac{7}{4}e^{-2t}-\frac{2}{3}e^{-t}-\frac{1}{12}e^{2t} \end{pmatrix}}$$

11. $\cancel{\begin{pmatrix}2\\-1\end{pmatrix} = x(0) = \Phi(0)\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}c_1\\c_2\end{pmatrix}}$

$$x(t) = \Phi(t)\begin{pmatrix}2\\-1\end{pmatrix} = \frac{7}{2}\begin{pmatrix}1\\1\end{pmatrix}e^{t} - \frac{3}{2}\begin{pmatrix}1\\3\end{pmatrix}e^{-t}.$$

14. $$\begin{cases} x' = Ax \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = \Phi(t)x_0$$

$$\Rightarrow \begin{cases} x' = Ax \\ x(t) = \Phi(t)x_0 \end{cases} \Rightarrow x(t+s) = \Phi(s)\Phi(t)x_0.$$

$$\begin{cases} x' = Ax \\ x(0) = x_0 \end{cases} \Rightarrow x(t+s) = \Phi(t+s)x_0.$$

By uniqueness, $\Phi(t)\Phi(s) = \Phi(t+s)$ $\square$

2.8. 1. $\det\begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_* = 1$ (repeated)

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} x_1 = 0 \Rightarrow x_1 = \begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} x_2 = \begin{pmatrix}2\\1\end{pmatrix} \Rightarrow x_2 = \begin{pmatrix}1\\0\end{pmatrix}.$$

$$x = c_1\begin{pmatrix}2\\1\end{pmatrix}e^{t} + c_2\left[\begin{pmatrix}2\\1\end{pmatrix}te^{t} + \begin{pmatrix}1\\0\end{pmatrix}e^{t}\right]$$

$|x| \to \infty$, asymptotic to $y=\frac{1}{2}x$, as $t\to\infty$

3. Similar.

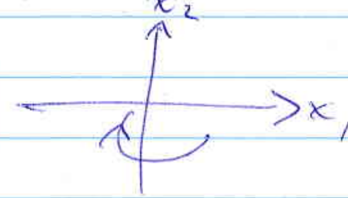
$$\boxed{x = c_1\begin{pmatrix}2\\1\end{pmatrix}e^{-t} + c_2\left[\begin{pmatrix}2\\1\end{pmatrix}te^{-t} + \begin{pmatrix}0\\2\end{pmatrix}e^{-t}\right]}$$

$\to 0$ as $t\to\infty$

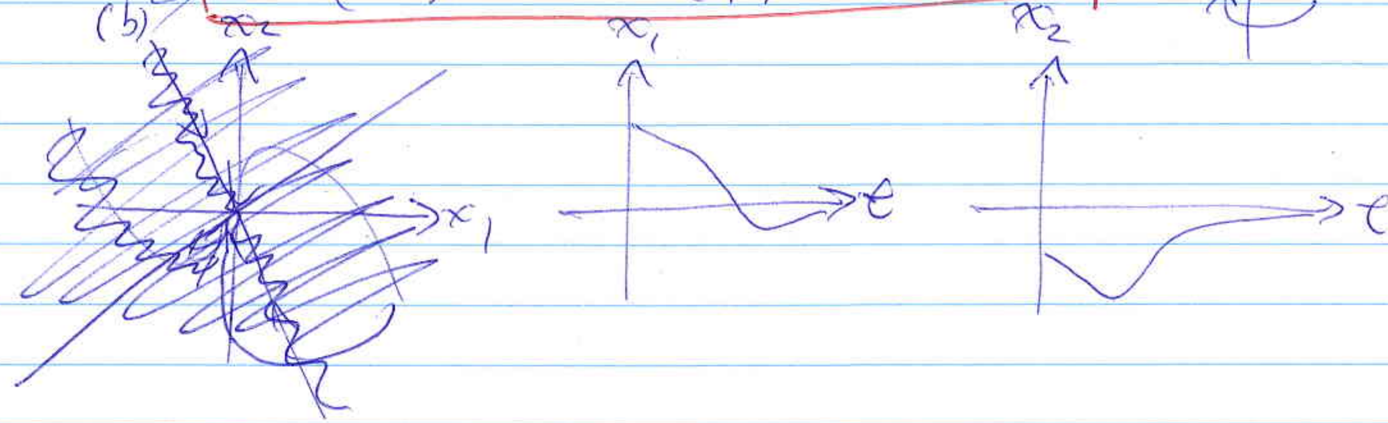
5. $$x = c_1\begin{pmatrix}-3\\4\\2\end{pmatrix}e^{-t} + c_2\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{2t} + c_3\left[\begin{pmatrix}0\\1\\-1\end{pmatrix}te^{2t} + \begin{pmatrix}1\\0\\1\end{pmatrix}e^{2t}\right]$$

6. $$x = c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-t} + c_3\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{-t}$$

8. (a) $$\boxed{x = \begin{pmatrix}3\\-1\end{pmatrix}e^{-t} - 6\begin{pmatrix}1\\1\end{pmatrix}te^{-t}}$$

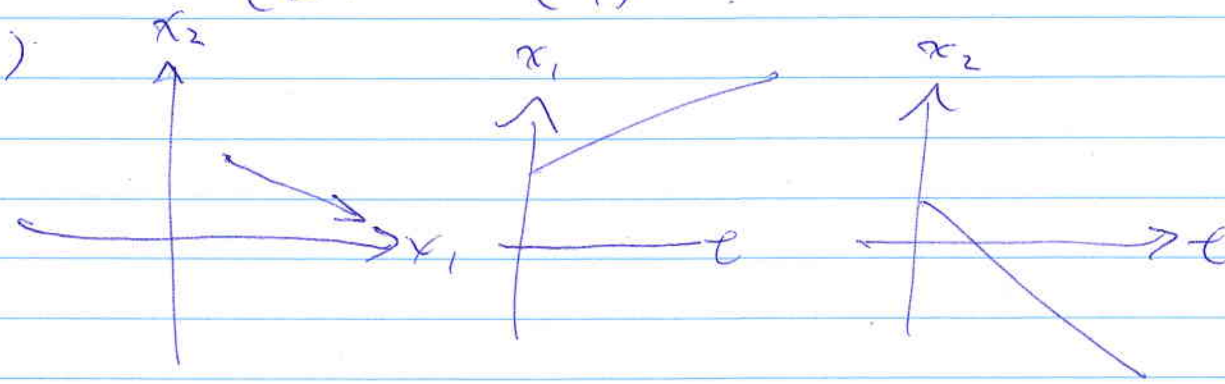


(b)



10. (a) $$x = 2\begin{pmatrix}1\\2\end{pmatrix} + 14\begin{pmatrix}3\\-1\end{pmatrix}t$$

(b)



13. $x = c_1\begin{pmatrix}2\\1\end{pmatrix}t + c_2\left[\begin{pmatrix}2\\1\end{pmatrix}t\log t + \begin{pmatrix}1\\0\end{pmatrix}t\right]$

(refer to 7.8 Q 1).

1.5. $\det\begin{pmatrix}a-\lambda & b\\ c & d-\lambda\end{pmatrix} = 0$ ~~$\lambda^2 + (ad-bc)\lambda$~~

$\iff \lambda^2 - (a+d)\lambda + (ad-bc) = 0$

$\iff \lambda = \dfrac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

$\iff \boxed{\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2 < 0}$ (both $= a+d < 0$ or both real < 0)

$$\begin{cases}\lambda_1+\lambda_2 = a+d < 0\\ \lambda_1\lambda_2 = ad-bc > 0\end{cases}$$

or i

$$\begin{cases}\lambda_1, \lambda_2 \in \mathbb{R} \Rightarrow \lambda_1, \lambda_2 < 0.\\ \lambda_1, \lambda_2 \notin \mathbb{R} \Rightarrow \mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2 < 0.\end{cases}$$

$x = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} \xrightarrow[t\to\infty]{} 0$.

1]

18. (a). $\det(A - \lambda I) = 0 \Rightarrow (\lambda-2)^3 = 0$

$$\begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 2 & 2\end{pmatrix}\xi^{(1)} = 0$$

$$\Rightarrow \xi^{(1)} = \begin{pmatrix}0\\1\\-1\end{pmatrix}. \quad (\xi_1^{(1)} = 0,\ \xi_2^{(1)} + \xi_3^{(1)} = 0)$$

(b). $x^{(1)}(t) = \begin{pmatrix}0\\1\\-1\end{pmatrix}e^{2t}$.

(c). $x = \xi t e^{2t} + \eta e^{2t}$.

$x' = \xi e^{2t} + 2\xi t e^{2t} + 2\eta e^{2t} = (\xi + 2\eta)e^{2t} + 2\xi t e^{2t}$.

$Ax = A\xi t e^{2t} + A\eta e^{2t}$

$$\Rightarrow \begin{cases}A\xi = 2\xi\\ A\eta = \xi + 2\eta.\end{cases} \Rightarrow \begin{cases}(A-2I)\xi = 0\\ (A-2I)\eta = \xi.\end{cases}$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \eta = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

$$\Rightarrow \eta = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$x^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

(d). Similar.

$$\zeta = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \quad \text{so} \quad x^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} t e^{2t} + \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} e^{2t}.$$

(e). $$\Phi(t) = e^{2t} \begin{pmatrix} 0 & 1 & t+2 \\ 1 & t+1 & \frac{1}{2}t^2 + t \\ -1 & -t & -\frac{1}{2}t^2 + 3 \end{pmatrix}.$$

(f). $$T = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}.$$

$$J = T^{-1} A T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

$\square$