7.9

1. $x' = \overbrace{\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}}^{A} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$.

$\det(A - \lambda I) = 0 \Rightarrow \lambda_{1,2} = \pm 1$.

$\lambda_1 = 1 \rightsquigarrow \xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -1 \rightsquigarrow \xi_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \right).$$

Let $x = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} u$.

$$u' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u + \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t \\ t \end{pmatrix}$$

i.e. $$\begin{cases} u_1' = u_1 + \frac{1}{2}(3e^t - t) \\ u_2' = -u_2 + \frac{1}{2}(-e^t + t) \end{cases}$$

Solving,

$$\begin{cases} u_1 = c_1 e^t + \frac{3}{2} t e^t + \frac{1}{2} t + \frac{1}{2}. \\ u_2 = c_2 e^{-t} - \frac{1}{4} e^t + \frac{1}{2} t - \frac{1}{2}. \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} x = {} & c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} \\ & + \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^t - \frac{1}{4} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}}$$

□.

Diagonalization

3. Fundamental matrix is

$$\Phi(t) = \begin{pmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & -\cos t + 2\sin t \end{pmatrix}.$$

$$x = \Phi(t)\left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \int^t \Phi(s)^{-1} \begin{pmatrix} -\cos s \\ \sin s \end{pmatrix} ds \right).$$

$$= c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}$$

$$+ \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} t\sin t - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t. \qquad \square$$

6. Similar.

$$x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t}$$

$$+ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \log t + \frac{8}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \frac{4}{25} \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \qquad \square$$

8. Homogeneous problem:

$$x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x$$

$$\Rightarrow x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

Let $x_p = (At+B)e^t$

$$x_p' = (At + A + B)e^t.$$

$$\Rightarrow \cancel{\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}} A te^t + (A+B)e^t = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} A te^t \cancel{+ \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} Be^t}$$

$$+ \left[\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix} B + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] e^t$$

$$\Rightarrow B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, A = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

$$\therefore \boxed{x = c_1\begin{pmatrix}1\\1\end{pmatrix}e^t + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t} + \begin{pmatrix}1\\0\end{pmatrix}e^t + \begin{pmatrix}2\\2\end{pmatrix}te^t}$$ □

undetermined coeff.

12. Similar.

$$\boxed{\begin{aligned} x = &\left[\frac{1}{5}\log(\sin t) - \log(\cos t) - \frac{2}{5}t + c_1\right]\begin{pmatrix}5\cos t\\ 2\cos t + \sin t\end{pmatrix} \\ &+ \left[\frac{2}{5}\log\cancel{(\sin t)} - \frac{4}{5}t + c_2\right]\begin{pmatrix}5\sin t\\ -\cos t + 2\sin t\end{pmatrix}.\end{aligned}}$$ □

variation of parameters

14. Similar.

$$\begin{aligned} x = &c_1\begin{pmatrix}1\\1\end{pmatrix}t + c_2\begin{pmatrix}1\\3\end{pmatrix}t^{-1} \\ &-\begin{pmatrix}2\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\3\end{pmatrix}t - \begin{pmatrix}1\\1\end{pmatrix}t\log t - \frac{1}{3}\begin{pmatrix}4\\3\end{pmatrix}t.\end{aligned}$$ □

6.1

5. (a) $\mathcal{L}(t) = \int_0^\infty te^{-st}\,dt = \dfrac{1}{s^2}$, $s>0$. □

(b) $\mathcal{L}(t^2) = \int_0^\infty t^2e^{-st}\,dt = \dfrac{2}{s^3}$, $s>0$. □

(c) $\mathcal{L}(t^n) = \int_0^\infty t^ne^{-st}\,dt = \dfrac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$

$$\overset{MI.}{=} \frac{n!}{s^{n+1}}, \quad s>0.$$ □

6. $\mathcal{L}(\cos at) = \int_0^\infty \cos at\, e^{-st}\,dt$

$$= \frac{s}{s^2+a^2}, \quad s>0$$ □

15. $\mathcal{L}(te^{at}) = \int_0^\infty te^{at}e^{-st}\,dt$

$$= \int_0^\infty te^{-(s-a)t}\,dt$$

$$= \boxed{\frac{1}{(s-a)^2}, \quad s>a}.$$

21. $\mathcal{L}(f) = \int_0^\pi e^{-st}\,dt = \boxed{\frac{1-e^{-\pi s}}{s}}$

23. $\mathcal{L}(f) = \int_0^1 te^{-st}\,dt + \int_1^\infty e^{-st}\,dt = \frac{1-e^{-s}}{s^2}$

6.2

1. $F(s) = \frac{3}{2}\frac{2}{s^2+2^2} \Rightarrow f(t) = \frac{3}{2}\sin 2t.$

3. $F(s) = \frac{2}{s^2+3s-4} = \frac{2}{5}\frac{(s+4)-(s-1)}{(s+4)(s-1)} = \frac{2}{5}\left(\frac{1}{s-1} - \frac{1}{s+4}\right)$

$$\Rightarrow f(t) = \frac{2}{5}(e^t - e^{-4t}).$$

9. $F(s) = \frac{1-2s}{s^2+4s+5} = \frac{5-2(s+2)}{(s+2)^2+1}$

$$\Rightarrow f(t) = \boxed{e^{-2t}(5\sin t - 2\cos t)}.$$

11. $\begin{cases} y''-y'-6y=0 \\ y(0)=1,\ y'(0)=-1 \end{cases}$

$$\Rightarrow (s^2Y-s+1)-(sY-1)-6Y=0.$$

$$\Rightarrow Y = \frac{s-2}{s^2-s-6} = \frac{4}{5}\frac{1}{s+2} + \frac{1}{5}\frac{1}{s-3}$$

$$\Rightarrow \boxed{y = \frac{4}{5}e^{-2t} + \frac{1}{5}e^{3t}},$$

$\square$

18. Similar.

$$\boxed{y = e^{t}\left(2\cos\sqrt{3}\,t - \frac{2}{\sqrt{3}}\sin\sqrt{3}\,t\right)},$$

$\square$