(20 points) 1. Solve the following ordinary differential equation

$$ty' + 2y = -4t^3y^3, \qquad y(1) = 1$$

and state the Interval of Existence.

<u>Solution</u>: This is Bernoulli's type equation

in which $n = 3$. —— 0

So let $v = y^{1-n} = y^{-2}$. —— 2

The original equation is —— 2

$$y' + \frac{2}{t}y = -4t^2y^3$$

So

$$\left.\begin{array}{l} v' + (1-3)\frac{2}{t}v = (1-3)(-4)t^2 \\ v' - \frac{4}{t}v = 8t^2 \end{array}\right\} 4$$

$$\left.\begin{array}{l} p = -\frac{4}{t}, \quad g = 8t^2 \\ \mu = e^{\int p} = e^{-\int \frac{4}{t}} = \frac{1}{t^4} \\ \int \mu g = \int \frac{1}{t^4} 8t^2 = \int \frac{8}{t^2} = -\frac{8}{t} \end{array}\right\} 4$$

$$v = \frac{1}{\mu}\left(c + \int \mu g\right) = t^4\left(c - \frac{8}{t}\right)$$

$t = 1,\ y = 1 \Rightarrow v = 1 \Rightarrow 1 = c - 8 \Rightarrow c = 9$ — 2

Thus $v = t^4\left(9 - \frac{8}{t}\right)$

$$y = \pm v^{-\frac{1}{2}} = \pm\left(\sqrt{t^4\left(9 - \frac{8}{t}\right)}\right)^{-\frac{1}{2}} = \left(\sqrt{t^3(9t-8)}\right)^{-\frac{1}{2}} \sqrt{t} \quad - 6$$

Interval of existence: 1) $t \neq 0$, $t \neq \frac{9}{8}$, $t^3(9t-8) > 0$, $t > 0$

2) equation $t \neq 0$

3) initial: $t_0 = 1$ —— 4

Interval of Existence: ~~$t > 0$~~ $t > \frac{8}{9}$

(20 points) 2. Solve the following ordinary differential equation

$$y' = \frac{t^2}{y^2 - 1}, \quad y(0) = 0$$

and state the Interval of Existence.

Solutions: This is separable — 4

$(y^2-1)\,y' = t^2$

$\frac{y^3}{3} - y = \frac{t^3}{3} + c$ — 8

$y^3 - 3y = t^3 + c$

$y(0)=0 \Rightarrow \quad y^3 - 3y = t^3$ — 2

Interval of Existence: 1) solution: for t.

2) equation: $y^2 \neq 1 \Leftrightarrow y = \pm 1$ — 2

$y=1 \Rightarrow \quad t = -\sqrt[3]{2}$

$y=-1 \Rightarrow \quad t = \sqrt[3]{2}$ — 2

3) initial condition: $t_0 = 0$

Thus interval of existace:

$-\sqrt[3]{2} < t < \sqrt[3]{2}$ — 2

(20 points) 3. Use the method of undetermined coefficients to find the general solution of

$$y'' - 6y' + 9y = te^{3t} - \sin(3t)$$

solution: Solve

$$\left.\begin{aligned} y''-6y'+9y &= te^{3t} \quad —(1) \\ y''-6y'+9y &= -\sin 3t \quad —(2) \end{aligned}\right\} \ 2$$

separately

homogeneous: $\left. r^2-6r+9=0 \Rightarrow r_1=r_2=3 \right\}$ — 2

$y_1 = e^{3t}, \ y_2 = te^{3t}$ — 2

$y_{p,1} = t^{s_1}(At+B)e^{3t}$ $\qquad y_{p,1} = (At^3+Bt^2)e^{3t}.$

$s_1 \neq 0, \ s_1 \neq 1 \Rightarrow s_1 = 2,$

$y'_{p,1} = (3At^2+2Bt)e^{3t} + 3(At^3+Bt^2)e^{3t}$

$= (3At^3 + (3A+3B)t^2 + 2Bt)e^{3t}$

$y''_{p,1} = (9At^2 + 2(3A+3B)t + 2B)e^{3t} + 3(3At^3 + (3A+3B)t^2 + 2Bt)e^{3t}$

$= (9At^3 + (18A+9B)t^2 + 2(3A+4B)t + 2B)e^{3t}$

$y''_{p,1} - 6y'_{p,1} + 9y_{p,1} = (2(3A+4B)t + 2B - 6\cdot 2Bt)e^{3t}$

$= ((6A-4B)t + 2B)e^{3t} = te^{3t}$ → 6

$2B = 0, \quad 6A - 4B = 1 \Rightarrow A = \frac{1}{6} \Rightarrow y_{p,1} = \frac{1}{6}t^3e^{3t}.$

— 2

$y_{p,2} = t^{s_2}(C\cos 3t + D\sin 3t)$ $\qquad s_2 = 0$

$y_{p,2} = C\cos 3t + D\sin 3t$

$y'_p = -3C\sin 3t + 3D\cos 3t$

$y''_p = -9C\cos 3t - 9D\sin 3t$

$y''_p - 6y'_p + 9y_p = (-9C + 18D + 9C)\cos 3t + (-9D - 18C + 9D)\sin 3t = -\sin 3t$

$= 18D\cos 3t - 18C\sin 3t$

$18D = 0, \quad -18C = -1 \Rightarrow C = \frac{1}{18} \Rightarrow y_{p,2} = \frac{1}{18}\cos 3t$ — 6

(40 points) 4. Consider the following differential equation

$$ty'' - (1+t)y' + y = 0, t > 0$$

(10 points) (a) Find the Wronskian $W(t)$.

(10 points) (b) Let $y_1(t) = e^t$. Use the reduction of order to find y_2.

(20 points) (c) Use the method of variation of parameters to solve the inhomogeneous problem

$$ty'' - (1+t)y' + y = t^2e^{-t}.$$

sol'ns: $y'' - \frac{1+t}{t}y' + \frac{1}{t}y = 0$, $p = -\frac{1+t}{t}$ — 2

(a) $W = e^{-\int p} = e^{\int \frac{1+t}{t}} = Cte^{t}$ — 8

(b) $y_1 = e^t$. Choose $W = te^t$, $y_2 = vy_1$

$v' = \frac{W}{y_1^2} = \frac{te^t}{e^{2t}} = te^{-t}$

$v = \int te^{-t} = -(t+1)e^{-t}$

$y_2 = vy_1 = -(t+1)$

10

(c) $y'' - \frac{1+t}{t}y' + \frac{1}{t}y = te^{-t}$ — 2

$g = te^{-t}$

$y_p = u_1y_1 + u_2y_2 \Rightarrow \begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g \end{cases}$ — 4

$\Rightarrow \begin{cases} u_1'e^t - u_2'(1+t) = 0 & (3) \\ u_1'e^t - u_2' = te^{-t} & (4) \end{cases}$ — 4

(3) − (4) $\Rightarrow -u_2't = -te^{-t} \Rightarrow u_2' = e^{-t} \Rightarrow u_2 = -e^{-t}$ — 4

$u_1' = e^{-t}u_2'(1+t) = -e^{-2t}(1+t) \Rightarrow u_1 = \frac{t}{2}e^{-2t} + \frac{3}{4}e^{-2t}$ — 4

$y_p = \left(\frac{t}{2} + \frac{3}{4}\right)e^{-t} + (1+t)e^{-t}$, $y = y_p + c_1y_1 + c_2y_2$

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