the result of an [illegible] understanding and interpreting [illegible] much more illuminating and helpful in understanding and interpreting a differential equation than a table of numbers or a complicated analytical formula. There are on the market several well-crafted and relatively inexpensive special-purpose software packages for the graphical investigation of differential equations. The widespread availability of personal computers has brought powerful computational and graphical capability within the reach of individual students. You should consider, in the light of your own circumstances, how best to take advantage of the available computing resources. You will surely find it enlightening to do so.

Another aspect of computer use that is very relevant to the study of differential equations is the availability of extremely powerful and general software packages that can perform a wide variety of mathematical operations. Among these are Maple, Mathematica, and MATLAB, each of which can be used on various kinds of personal computers or workstations. All three of these packages can execute extensive numerical computations and have versatile graphical facilities. Maple and Mathematica also have very extensive analytical capabilities. For example, they can perform the analytical steps involved in solving many differential equations, often in response to a single command. Anyone who expects to deal with differential equations in more than a superficial way should become familiar with at least one of these products and explore the ways in which it can be used.

For you, the student, these computing resources have an effect on how you should study differential equations. To become confident in using differential equations, it is essential to understand how the solution methods work, and this understanding is achieved, in part, by working out a sufficient number of examples in detail. However, eventually you should plan to delegate as many as possible of the routine (often repetitive) details to a computer, while you focus on the proper formulation of the problem and on the interpretation of the solution. Our viewpoint is that you should always try to use the best methods and tools available for each task. In particular, you should strive to combine numerical, graphical, and analytical methods so as to attain maximum understanding of the behavior of the solution and of the underlying process that the problem models. You should also remember that some tasks can best be done with pencil and paper, while others require a calculator or computer. Good judgment is often needed in selecting an effective combination.

1.3

PROBLEMS

In each of Problems 1 through 6, determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $t^2 \dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y = \sin t$

2. $(1 + y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$

3. $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$

4. $\dfrac{dy}{dt} + ty^2 = 0$

5. $\dfrac{d^2y}{dt^2} + \sin(t + y) = \sin t$

6. $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2 t)y = t^3$

In each of Problems 7 through 14, verify that each given function is a solution of the differential equation.

7. $y'' - y = 0; \quad y_1(t) = e^t, \quad y_2(t) = \cosh t$

8. $y'' + 2y' - 3y = 0; \quad y_1(t) = e^{-3t}, \quad y_2(t) = e^t$
9. $ty' - y = t^2; \quad y = 3t + t^2$
10. $y'''' + 4y''' + 3y = t; \quad y_1(t) = t/3, \quad y_2(t) = e^{-t} + t/3$
11. $2t^2y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}$
12. $t^2y'' + 5ty' + 4y = 0, \quad t > 0; \quad y_1(t) = t^{-2}, \quad y_2(t) = t^{-2}\ln t$
13. $y'' + y = \sec t, \quad 0 < t < \pi/2; \quad y = (\cos t)\ln\cos t + t\sin t$
14. $y' - 2ty = 1; \quad y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$

In each of Problems 15 through 18, determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

15. $y' + 2y = 0$
16. $y'' - y = 0$
17. $y'' + y' - 6y = 0$
18. $y''' - 3y'' + 2y' = 0$

In each of Problems 19 and 20, determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

19. $t^2y'' + 4ty' + 2y = 0$
20. $t^2y'' - 4ty' + 4y = 0$

In each of Problems 21 through 24, determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.

21. $u_{xx} + u_{yy} + u_{zz} = 0$
22. $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$
23. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$
24. $u_t + uu_x = 1 + u_{xx}$

In each of Problems 25 through 28, verify that each given function is a solution of the given partial differential equation.

25. $u_{xx} + u_{yy} = 0; \quad u_1(x,y) = \cos x\cosh y, \quad u_2(x,y) = \ln(x^2 + y^2)$
26. $\alpha^2 u_{xx} = u_t; \quad u_1(x,t) = e^{-\alpha^2 t}\sin x, \quad u_2(x,t) = e^{-\alpha^2\lambda^2 t}\sin\lambda x, \quad \lambda$ a real constant
27. $a^2 u_{xx} = u_{tt}; \quad u_1(x,t) = \sin\lambda x\sin\lambda at, \quad u_2(x,t) = \sin(x - at), \quad \lambda$ a real constant
28. $\alpha^2 u_{xx} = u_t; \quad u = (\pi/t)^{1/2}e^{-x^2/4\alpha^2 t}, \quad t > 0$
29. Follow the steps indicated here to derive the equation of motion of a pendulum, Eq. (12) in the text. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.
    (a) Assume that the mass is in an arbitrary displaced position, indicated by the angle θ. Draw a free-body diagram showing the forces acting on the mass.
    (b) Apply Newton's law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is $Ld^2\theta/dt^2$, where L is the length of the rod.
    (c) Simplify the result from part (b) to obtain Eq. (12) in the text.
30. Another way to derive the pendulum equation (12) is based on the principle of conservation of energy.
    (a) Show that the kinetic energy T of the pendulum in motion is
$$T = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2.$$
    (b) Show that the potential energy V of the pendulum, relative to its rest position, is
$$V = mgL(1 - \cos\theta).$$

1. $y' + 3y = t + e^{-2t}$
2. $y' - 2y = t^2 e^{2t}$
3. $y' + y = te^{-t} + 1$
4. $y' + (1/t)y = 3\cos 2t, \quad t > 0$
5. $y' - 2y = 3e^t$
6. $ty' + 2y = \sin t, \quad t > 0$
7. $y' + 2ty = 2te^{-t^2}$
8. $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$
9. $2y' + y = 3t$
10. $ty' - y = t^2 e^{-t}, \quad t > 0$
11. $y' + y = 5\sin 2t$
12. $2y' + y = 3t^2$

In each of Problems 13 through 20, find the solution of the given initial value problem.

13. $y' - y = 2te^{2t}, \quad y(0) = 1$
14. $y' + 2y = te^{-2t}, \quad y(1) = 0$
15. $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$
16. $y' + (2/t)y = (\cos t)/t^2, \quad y(\pi) = 0, \quad t > 0$
17. $y' - 2y = e^{2t}, \quad y(0) = 2$
18. $ty' + 2y = \sin t, \quad y(\pi/2) = 1, \quad t > 0$
19. $t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0, \quad t < 0$
20. $ty' + (t + 1)y = t, \quad y(\ln 2) = 1, \quad t > 0$

In each of Problems 21 through 23:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

(b) Solve the initial value problem and find the critical value a_0 exactly.

(c) Describe the behavior of the solution corresponding to the initial value a_0.

21. $y' - \frac{1}{2}y = 2\cos t, \quad y(0) = a$
22. $2y' - y = e^{t/3}, \quad y(0) = a$
23. $3y' - 2y = e^{-\pi t/2}, \quad y(0) = a$

In each of Problems 24 through 26:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

(b) Solve the initial value problem and find the critical value a_0 exactly.

(c) Describe the behavior of the solution corresponding to the initial value a_0.

24. $ty' + (t + 1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0$
25. $ty' + 2y = (\sin t)/t, \quad y(-\pi/2) = a, \quad t < 0$
26. $(\sin t)y' + (\cos t)y = e^t, \quad y(1) = a, \quad 0 < t < \pi$
27. Consider the initial value problem

$$y' + \tfrac{1}{2}y = 2\cos t, \qquad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

28. Consider the initial value problem

$$y' + \tfrac{2}{3}y = 1 - \tfrac{1}{2}t, \qquad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

29. Consider the initial value problem

$$y' + \tfrac{1}{4}y = 3 + 2\cos 2t, \qquad y(0) = 0.$$

(a) Find the solution of this initial value problem and describe its behavior for large t.
(b) Determine the value of t for which the solution first intersects the line $y = 12$.

30. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

31. Consider the initial value problem

$$y' - \tfrac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

32. Show that all solutions of $2y' + ty = 2$ [Eq. (41) of the text] approach a limit as $t \to \infty$, and find the limiting value.
Hint: Consider the general solution, Eq. (47), and use L'Hôpital's rule on the first term.

33. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 34 through 37, construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

34. All solutions have the limit 3 as $t \to \infty$.
35. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.
36. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.
37. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.
38. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \qquad \text{(i)}$$

(a) If $g(t) = 0$ for all t, show that the solution is

$$y = A\exp\left[-\int p(t)\,dt\right], \qquad \text{(ii)}$$

where A is a constant.
(b) If $g(t)$ is not everywhere zero, assume that the solution of Eq. (i) is of the form

$$y = A(t)\exp\left[-\int p(t)\,dt\right], \qquad \text{(iii)}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t) \exp\left[\int p(t)\,dt\right]. \qquad \text{(iv)}$$

(c) Find $A(t)$ from Eq. (iv). Then substitute for $A(t)$ in Eq. (iii) and determine y. Verify that the solution obtained in this manner agrees with that of Eq. (33) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.6 in connection with second order linear equations.

In each of Problems 39 through 42, use the method of Problem 38 to solve the given differential equation.

39. $y' - 2y = t^2 e^{2t}$
40. $y' + (1/t)y = 3\cos 2t, \quad t > 0$
41. $ty' + 2y = \sin t, \quad t > 0$
42. $2y' + y = 3t^2$

2.2 Separable Equations

In Section 1.2 we used a process of direct integration to solve first order linear equations of the form

$$\frac{dy}{dt} = ay + b, \qquad (1)$$

where a and b are constants. We will now show that this process is actually applicable to a much larger class of equations.

We will use x, rather than t, to denote the independent variable in this section for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, x often occurs as the independent variable. Further, we want to reserve t for another purpose later in the section.

The general first order equation is

$$\frac{dy}{dx} = f(x, y). \qquad (2)$$

Linear equations were considered in the preceding section, but if Eq. (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first order equations that can be solved by direct integration.

To identify this class of equations, we first rewrite Eq. (2) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \qquad (3)$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, but there may be other ways as well. If it happens that M is a function of x only and N is a function of y only, then Eq. (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \qquad (4)$$

then, by comparing numerators and denominators in Eqs. (26) and (27), we obtain the system

$$dx/dt = G(x, y), \qquad dy/dt = F(x, y). \tag{28}$$

At first sight it may seem unlikely that a problem will be simplified by replacing a single equation by a pair of equations, but in fact, the system (28) may well be more amenable to investigation than the single equation (27). Chapter 9 is devoted to nonlinear systems of the form (28).

Note 3: In Example 2 it was not difficult to solve explicitly for y as a function of x. However, this situation is exceptional, and often it will be better to leave the solution in implicit form, as in Examples 1 and 3. Thus, in the problems below and in other sections where nonlinear equations appear, the words "solve the following differential equation" mean to find the solution explicitly if it is convenient to do so, but otherwise to find an equation defining the solution implicitly.

2.2

PROBLEMS

In each of Problems 1 through 8, solve the given differential equation.

1. $y' = x^2/y$
2. $y' = x^2/y(1+x^3)$
3. $y' + y^2 \sin x = 0$
4. $y' = (3x^2 - 1)/(3 + 2y)$
5. $y' = (\cos^2 x)(\cos^2 2y)$
6. $xy' = (1 - y^2)^{1/2}$
7. $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$
8. $\dfrac{dy}{dx} = \dfrac{x^2}{1 + y^2}$

In each of Problems 9 through 20:
(a) Find the solution of the given initial value problem in explicit form.
(b) Plot the graph of the solution.
(c) Determine (at least approximately) the interval in which the solution is defined.

9. $y' = (1 - 2x)y^2, \quad y(0) = -1/6$
10. $y' = (1 - 2x)/y, \quad y(1) = -2$
11. $x\,dx + ye^{-x}dy = 0, \quad y(0) = 1$
12. $dr/d\theta = r^2/\theta, \quad r(1) = 2$
13. $y' = 2x/(y + x^2y), \quad y(0) = -2$
14. $y' = xy^3(1 + x^2)^{-1/2}, \quad y(0) = 1$
15. $y' = 2x/(1 + 2y), \quad y(2) = 0$
16. $y' = x(x^2 + 1)/4y^3, \quad y(0) = -1/\sqrt{2}$
17. $y' = (3x^2 - e^x)/(2y - 5), \quad y(0) = 1$
18. $y' = (e^{-x} - e^x)/(3 + 4y), \quad y(0) = 1$
19. $\sin 2x\,dx + \cos 3y\,dy = 0, \quad y(\pi/2) = \pi/3$
20. $y^2(1 - x^2)^{1/2}dy = \arcsin x\,dx, \quad y(0) = 1$

Some of the results requested in Problems 21 through 28 can be obtained either by solving the given equations analytically or by plotting numerically generated approximations to the solutions. Try to form an opinion about the advantages and disadvantages of each approach.

21. Solve the initial value problem

$$y' = (1 + 3x^2)/(3y^2 - 6y), \qquad y(0) = 1$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

22. Solve the initial value problem

$$y' = 3x^2/(3y^2 - 4), \qquad y(1) = 0$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

23. Solve the initial value problem

$$y' = 2y^2 + xy^2, \qquad y(0) = 1$$

and determine where the solution attains its minimum value.

24. Solve the initial value problem

$$y' = (2 - e^x)/(3 + 2y), \qquad y(0) = 0$$

and determine where the solution attains its maximum value.

25. Solve the initial value problem

$$y' = 2\cos 2x/(3 + 2y), \qquad y(0) = -1$$

and determine where the solution attains its maximum value.

26. Solve the initial value problem

$$y' = 2(1 + x)(1 + y^2), \qquad y(0) = 0$$

and determine where the solution attains its minimum value.

27. Consider the initial value problem

$$y' = ty(4 - y)/3, \qquad y(0) = y_0.$$

(a) Determine how the behavior of the solution as t increases depends on the initial value y_0.
(b) Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

28. Consider the initial value problem

$$y' = ty(4 - y)/(1 + t), \qquad y(0) = y_0 > 0.$$

(a) Determine how the solution behaves as $t \to \infty$.
(b) If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.
(c) Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

29. Solve the equation

$$\frac{dy}{dx} = \frac{ay + b}{cy + d},$$

where a, b, c, and d are constants.

Homogeneous Equations. If the right side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the ratio y/x only, then the equation is said to be homogeneous.[1] Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 30 illustrates how to solve first order homogeneous equations.

30. Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}. \qquad \text{(i)}$$

(a) Show that Eq. (i) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \qquad \text{(ii)}$$

thus Eq. (i) is homogeneous.

(b) Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.

(c) Replace y and dy/dx in Eq. (ii) by the expressions from part (b) that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \qquad \text{(iii)}$$

Observe that Eq. (iii) is separable.

(d) Solve Eq. (iii), obtaining v implicitly in terms of x.

(e) Find the solution of Eq. (i) by replacing v by y/x in the solution in part (d).

(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (i) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore, the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 31 through 38:

(a) Show that the given equation is homogeneous.

(b) Solve the differential equation.

(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

31. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$

32. $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$

33. $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$

34. $\dfrac{dy}{dx} = -\dfrac{4x + 3y}{2x + y}$

[1] The word "homogeneous" has different meanings in different mathematical contexts. The homogeneous equations considered here have nothing to do with the homogeneous equations that will occur in Chapter 3 and elsewhere.