3.7

20. Assume that the system described by the equation $mu'' + \gamma u' + ku = 0$ is critically damped and that the initial conditions are $u(0) = u_0, u'(0) = v_0$. If $v_0 = 0$, show that $u \to 0$ as $t \to \infty$ but that u is never zero. If u_0 is positive, determine a condition on v_0 that will ensure that the mass passes through its equilibrium position after it is released.

21. **Logarithmic Decrement.** (a) For the damped oscillation described by Eq. (26), show that the time between successive maxima is $T_d = 2\pi/\mu$.

(b) Show that the ratio of the displacements at two successive maxima is given by $\exp(\gamma T_d/2m)$. Observe that this ratio does not depend on which pair of maxima is chosen. The natural logarithm of this ratio is called the logarithmic decrement and is denoted by Δ.

(c) Show that $\Delta = \pi\gamma/m\mu$. Since m, μ, and Δ are quantities that can be measured easily for a mechanical system, this result provides a convenient and *practical* method for determining the damping constant of the system, which is more difficult to measure directly. In particular, for the motion of a vibrating mass in a viscous fluid, the damping constant depends on the viscosity of the fluid; for simple geometric shapes the form of this dependence is known, and the preceding relation allows the experimental determination of the viscosity. This is one of the most accurate ways of determining the viscosity of a gas at high pressure.

22. Referring to Problem 21, find the logarithmic decrement of the system in Problem 10.

23. For the system in Problem 17, suppose that $\Delta = 3$ and $T_d = 0.3$ s. Referring to Problem 21, determine the value of the damping coefficient γ.

24. The position of a certain spring–mass system satisfies the initial value problem

$$\tfrac{3}{2}u'' + ku = 0, \qquad u(0) = 2, \quad u'(0) = v.$$

If the period and amplitude of the resulting motion are observed to be π and 3, respectively, determine the values of k and v.

25. Consider the initial value problem

$$u'' + \gamma u' + u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$

We wish to explore how long a time interval is required for the solution to become "negligible" and how this interval depends on the damping coefficient γ. To be more precise, let us seek the time τ such that $|u(t)| < 0.01$ for all $t > \tau$. Note that critical damping for this problem occurs for $\gamma = 2$.

(a) Let $\gamma = 0.25$ and determine τ, or at least estimate it fairly accurately from a plot of the solution.

(b) Repeat part (a) for several other values of γ in the interval $0 < \gamma < 1.5$. Note that τ steadily decreases as γ increases for γ in this range.

(c) Create a graph of τ versus γ by plotting the pairs of values found in parts (a) and (b). Is the graph a smooth curve?

(d) Repeat part (b) for values of γ between 1.5 and 2. Show that τ continues to decrease until γ reaches a certain critical value γ_0, after which τ increases. Find γ_0 and the corresponding minimum value of τ to two decimal places.

(e) Another way to proceed is to write the solution of the initial value problem in the form (26). Neglect the cosine factor and consider only the exponential factor and the amplitude R. Then find an expression for τ as a function of γ. Compare the approximate results obtained in this way with the values determined in parts (a), (b), and (d).

26. Consider the initial value problem

$$mu'' + \gamma u' + ku = 0, \qquad u(0) = u_0, \quad u'(0) = v_0.$$

Assume that $\gamma^2 < 4km$.

(a) Solve the initial value problem.

(b) Write the solution in the form $u(t) = R\exp(-\gamma t/2m)\cos(\mu t - \delta)$. Determine R in terms of m, γ, k, u_0, and v_0.

(c) Investigate the dependence of R on the damping coefficient γ for fixed values of the other parameters.

27. A cubic block of side l and mass density ρ per unit volume is floating in a fluid of mass density ρ_0 per unit volume, where $\rho_0 > \rho$. If the block is slightly depressed and then released, it oscillates in the vertical direction. Assuming that the viscous damping of the fluid and air can be neglected, derive the differential equation of motion and determine the period of the motion.
Hint: Use Archimedes'[10] principle: an object that is completely or partially submerged in a fluid is acted on by an upward (buoyant) force equal to the weight of the displaced fluid.

28. The position of a certain undamped spring–mass system satisfies the initial value problem

$$u'' + 2u = 0, \qquad u(0) = 0, \quad u'(0) = 2.$$

(a) Find the solution of this initial value problem.

(b) Plot u versus t and u' versus t on the same axes.

(c) Plot u' versus u; that is, plot $u(t)$ and $u'(t)$ parametrically with t as the parameter. This plot is known as a phase plot, and the uu'-plane is called the phase plane. Observe that a closed curve in the phase plane corresponds to a periodic solution $u(t)$. What is the direction of motion on the phase plot as t increases?

29. The position of a certain spring–mass system satisfies the initial value problem

$$u'' + \tfrac{1}{4}u' + 2u = 0, \qquad u(0) = 0, \quad u'(0) = 2.$$

(a) Find the solution of this initial value problem.

(b) Plot u versus t and u' versus t on the same axes.

(c) Plot u' versus u in the phase plane (see Problem 28). Identify several corresponding points on the curves in parts (b) and (c). What is the direction of motion on the phase plot as t increases?

30. In the absence of damping, the motion of a spring–mass system satisfies the initial value problem

$$mu'' + ku = 0, \qquad u(0) = a, \quad u'(0) = b.$$

(a) Show that the kinetic energy initially imparted to the mass is $mb^2/2$ and that the potential energy initially stored in the spring is $ka^2/2$, so that initially the total energy in the system is $(ka^2 + mb^2)/2$.

(b) Solve the given initial value problem.

(c) Using the solution in part (b), determine the total energy in the system at any time. Your result should confirm the principle of conservation of energy for this system.

[10] Archimedes (287–212 BC) was the foremost of the ancient Greek mathematicians. He lived in Syracuse on the island of Sicily. His most notable discoveries were in geometry, but he also made important contributions to hydrostatics and other branches of mechanics. His method of exhaustion is a precursor of the integral calculus developed by Newton and Leibniz almost two millennia later. He died at the hands of a Roman soldier during the Second Punic War.

3.8

15. Find the solution of the initial value problem

$$u'' + u = F(t), \qquad u(0) = 0, \quad u'(0) = 0,$$

where

$$F(t) = \begin{cases} F_0 t, & 0 \le t \le \pi, \\ F_0(2\pi - t), & \pi < t \le 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

Hint: Treat each time interval separately, and match the solutions in the different intervals by requiring u and u' to be continuous functions of t.

16. A series circuit has a capacitor of 0.25×10^{-6} F, a resistor of 5×10^3 Ω, and an inductor of 1 H. The initial charge on the capacitor is zero. If a 12-volt battery is connected to the circuit and the circuit is closed at $t = 0$, determine the charge on the capacitor at $t = 0.001$ s, at $t = 0.01$ s, and at any time t. Also determine the limiting charge as $t \to \infty$.

17. Consider a vibrating system described by the initial value problem

$$u'' + \tfrac{1}{4}u' + 2u = 2\cos \omega t, \qquad u(0) = 0, \quad u'(0) = 2.$$

(a) Determine the steady state part of the solution of this problem.
(b) Find the amplitude A of the steady state solution in terms of ω.
(c) Plot A versus ω.
(d) Find the maximum value of A and the frequency ω for which it occurs.

18. Consider the forced but undamped system described by the initial value problem

$$u'' + u = 3\cos \omega t, \qquad u(0) = 0, \quad u'(0) = 0.$$

(a) Find the solution $u(t)$ for $\omega \neq 1$.
(b) Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Describe how the response $u(t)$ changes as ω varies in this interval. What happens as ω takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

19. Consider the vibrating system described by the initial value problem

$$u'' + u = 3\cos \omega t, \qquad u(0) = 1, \quad u'(0) = 1.$$

(a) Find the solution for $\omega \neq 1$.
(b) Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Compare the results with those of Problem 18; that is, describe the effect of the nonzero initial conditions.

20. For the initial value problem in Problem 18, plot u' versus u for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Such a plot is called a phase plot. Use a t interval that is long enough so that the phase plot appears as a closed curve. Mark your curve with arrows to show the direction in which it is traversed as t increases.

Problems 21 through 23 deal with the initial value problem

$$u'' + 0.125u' + 4u = F(t), \qquad u(0) = 2, \quad u'(0) = 0.$$

In each of these problems:

(a) Plot the given forcing function $F(t)$ versus t, and also plot the solution $u(t)$ versus t on the same set of axes. Use a t interval that is long enough so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that $\omega_0 = \sqrt{k/m} = 2$.

(b) Draw the phase plot of the solution; that is, plot u' versus u.

we could have chosen $\mathbf{x}^{(2)}$ as before and $\mathbf{x}^{(3)}$ by using $c_1 = 1$ and $c_2 = -2$ in E
In this way we obtain

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

as the eigenvectors associated with the eigenvalue $\lambda = -1$. These eigenve orthogonal to each other as well as to the eigenvector $\mathbf{x}^{(1)}$ that correspo eigenvalue $\lambda = 2$.

7.3

PROBLEMS

In each of Problems 1 through 6, either solve the given system of equations, or else there is no solution.

1. $$\begin{aligned} x_1 \qquad - x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 1 \\ -x_1 + x_2 + 2x_3 &= 2 \end{aligned}$$

2. $$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 1 \end{aligned}$$

3. $$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 - x_2 - 2x_3 &= -1 \end{aligned}$$

4. $$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

5. $$\begin{aligned} x_1 \qquad - x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x_1 + 2x_2 - x_3 &= -2 \\ -2x_1 - 4x_2 + 2x_3 &= 4 \\ 2x_1 + 4x_2 - 2x_3 &= -4 \end{aligned}$$

In each of Problems 7 through 11, determine whether the members of the given se are linearly independent. If they are linearly dependent, find a linear relation a The vectors are written as row vectors to save space but may be considered as col that is, the transposes of the given vectors may be used instead of the vectors the

7. $\mathbf{x}^{(1)} = (1, 1, 0)$, $\quad \mathbf{x}^{(2)} = (0, 1, 1)$, $\quad \mathbf{x}^{(3)} = (1, 0, 1)$
8. $\mathbf{x}^{(1)} = (2, 1, 0)$, $\quad \mathbf{x}^{(2)} = (0, 1, 0)$, $\quad \mathbf{x}^{(3)} = (-1, 2, 0)$
9. $\mathbf{x}^{(1)} = (1, 2, 2, 3)$, $\quad \mathbf{x}^{(2)} = (-1, 0, 3, 1)$, $\quad \mathbf{x}^{(3)} = (-2, -1, 1, 0)$, $\quad \mathbf{x}^{(4)} = -$
10. $\mathbf{x}^{(1)} = (1, 2, -1, 0)$, $\quad \mathbf{x}^{(2)} = (2, 3, 1, -1)$, $\quad \mathbf{x}^{(3)} = (-1, 0, 2, 2)$, $\quad \mathbf{x}^{(4)} =$
11. $\mathbf{x}^{(1)} = (1, 2, -2)$, $\quad \mathbf{x}^{(2)} = (3, 1, 0)$, $\quad \mathbf{x}^{(3)} = (2, -1, 1)$, $\quad \mathbf{x}^{(4)} = (4, 3, -2$
12. Suppose that each of the vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ has n components, where $n <$ $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent.

In each of Problems 13 and 14, determine whether the members of the given set of linearly independent for $-\infty < t < \infty$. If they are linearly dependent, find the line among them. As in Problems 7 through 11, the vectors are written as row vectors to

13. $\mathbf{x}^{(1)}(t) = (e^{-t}, 2e^{-t})$, $\quad \mathbf{x}^{(2)}(t) = (e^{-t}, e^{-t})$, $\quad \mathbf{x}^{(3)}(t) = (3e^{-t}, 0)$
14. $\mathbf{x}^{(1)}(t) = (2\sin t, \sin t)$, $\quad \mathbf{x}^{(2)}(t) = (\sin t, 2\sin t)$

15. Let

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

Show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent at each point in the interval $0 \le t \le 1$. Nevertheless, show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on $0 \le t \le 1$.

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

16. $\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

17. $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

18. $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

19. $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

20. $\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

21. $\begin{pmatrix} -3 & 3/4 \\ -5 & 1 \end{pmatrix}$

22. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$

23. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

24. $\begin{pmatrix} 11/9 & -2/9 & 8/9 \\ -2/9 & 2/9 & 10/9 \\ 8/9 & 10/9 & 5/9 \end{pmatrix}$

25. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

Problems 26 through 30 deal with the problem of solving $\mathbf{Ax} = \mathbf{b}$ when $\det \mathbf{A} = 0$.

26. (a) Suppose that $\mathbf{A}$ is a real-valued $n \times n$ matrix. Show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^T\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.
Hint: You may find it simpler to consider first the case $n = 2$; then extend the result to an arbitrary value of n.
(b) If $\mathbf{A}$ is not necessarily real, show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^*\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.
(c) If $\mathbf{A}$ is Hermitian, show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{Ay})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.

27. Suppose that, for a given matrix $\mathbf{A}$, there is a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$. Show that there is also a nonzero vector $\mathbf{y}$ such that $\mathbf{A}^*\mathbf{y} = \mathbf{0}$.

28. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{Ax} = \mathbf{b}$ has solutions. Show that $(\mathbf{b}, \mathbf{y}) = 0$, where $\mathbf{y}$ is any solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Verify that this statement is true for the set of equations in Example 2.
Hint: Use the result of Problem 26(b).

29. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{x} = \mathbf{x}^{(0)}$ is a solution of $\mathbf{Ax} = \mathbf{b}$. Show that if ξ is a solution of $\mathbf{A}\xi = \mathbf{0}$ and α is any constant, then $\mathbf{x} = \mathbf{x}^{(0)} + \alpha\xi$ is also a solution of $\mathbf{Ax} = \mathbf{b}$.

30. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{y}$ is a solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Show that if $(\mathbf{b}, \mathbf{y}) = 0$ for every such $\mathbf{y}$, then $\mathbf{Ax} = \mathbf{b}$ has solutions. Note that this is the converse of Problem 28; the form of the solution is given by Problem 29.
Hint: What does the relation $\mathbf{A}^*\mathbf{y} = \mathbf{0}$ say about the rows of $\mathbf{A}$? Again, it may be helpful to consider the case $n = 2$ first.

To prove this theorem, note that the existence and uniqueness of the $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ mentioned in Theorem 7.4.4 are ensured by Theorem 7.1. hard to see that the Wronskian of these solutions is equal to 1 when $t = t_0$; $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are a fundamental set of solutions.

Once one fundamental set of solutions has been found, other sets can ated by forming (independent) linear combinations of the first set. For th purposes, the set given by Theorem 7.4.4 is usually the simplest.

Finally, it may happen (just as for second order linear equations) tha whose coefficients are all real may give rise to solutions that are comple In this case, the following theorem is analogous to Theorem 3.2.6 and ena obtain real-valued solutions instead.

Theorem 7.4.5 Consider the system (3)

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x},$$

where each element of $\mathbf{P}$ is a real-valued continuous function. If $\mathbf{x} = \mathbf{u}(t)$ – a complex-valued solution of Eq. (3), then its real part $\mathbf{u}(t)$ and its imagin $\mathbf{v}(t)$ are also solutions of this equation.

To prove this result, we substitute $\mathbf{u}(t) + i\mathbf{v}(t)$ for $\mathbf{x}$ in Eq. (3), thereby ob

$$\mathbf{x}' - \mathbf{P}(t)\mathbf{x} = \mathbf{u}'(t) - \mathbf{P}(t)\mathbf{u}(t) + i\,[\mathbf{v}'(t) - \mathbf{P}(t)\mathbf{v}(t)] = \mathbf{0}.$$

We have used the assumption that $\mathbf{P}(t)$ is real-valued to separate Eq. its real and imaginary parts. Since a complex number is zero if and o real and imaginary parts are both zero, we conclude that $\mathbf{u}'(t) - \mathbf{P}(t)\mathbf{u}(t) =$ $\mathbf{v}'(t) - \mathbf{P}(t)\mathbf{v}(t) = \mathbf{0}$. Therefore, $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions of Eq. (3).

To summarize the results of this section:

1. Any set of n linearly independent solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ con fundamental set of solutions.
2. Under the conditions given in this section, such fundamental sets always exist.
3. Every solution of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ can be represented as a linear combina fundamental set of solutions.

PROBLEMS

1. Prove the generalization of Theorem 7.4.1, as expressed in the sentence tha Eq. (8), for an arbitrary value of the integer k.
2. In this problem we outline a proof of Theorem 7.4.3 in the case $n = 2$. Let $\mathbf{x}$ be solutions of Eq. (3) for $\alpha < t < \beta$, and let W be the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}$
   (a) Show that

$$\frac{dW}{dt} = \begin{vmatrix} \dfrac{dx_1^{(1)}}{dt} & \dfrac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \dfrac{dx_2^{(1)}}{dt} & \dfrac{dx_2^{(2)}}{dt} \end{vmatrix}.$$

(b) Using Eq. (3), show that

$$\frac{dW}{dt} = (p_{11} + p_{22})W.$$

(c) Find $W(t)$ by solving the differential equation obtained in part (b). Use this expression to obtain the conclusion stated in Theorem 7.4.3.

(d) Prove Theorem 7.4.3 for an arbitrary value of n by generalizing the procedure of parts (a), (b), and (c).

3. Show that the Wronskians of two fundamental sets of solutions of the system (3) can differ at most by a multiplicative constant.
Hint: Use Eq. (15).

4. If $x_1 = y$ and $x_2 = y'$, then the second order equation

$$y'' + p(t)y' + q(t)y = 0 \tag{i}$$

corresponds to the system

$$x_1' = x_2,$$
$$x_2' = -q(t)x_1 - p(t)x_2. \tag{ii}$$

Show that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are a fundamental set of solutions of Eqs. (ii), and if $y^{(1)}$ and $y^{(2)}$ are a fundamental set of solutions of Eq. (i), then $W[y^{(1)}, y^{(2)}] = cW[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}]$, where c is a nonzero constant.
Hint: $y^{(1)}(t)$ and $y^{(2)}(t)$ must be linear combinations of $x_{11}(t)$ and $x_{12}(t)$.

5. Show that the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the sum of any particular solution $\mathbf{x}^{(p)}$ of this equation and the general solution $\mathbf{x}^{(c)}$ of the corresponding homogeneous equation.

6. Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$.

(a) Compute the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

(b) In what intervals are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?

(c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?

(d) Find this system of equations and verify the conclusions of part (c).

7. Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$, and answer the same questions as in Problem 6.

The following two problems indicate an alternative derivation of Theorem 7.4.2.

8. Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on the interval $\alpha < t < \beta$. Assume that $\mathbf{P}$ is continuous, and let t_0 be an arbitrary point in the given interval. Show that $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are linearly dependent for $\alpha < t < \beta$ if (and only if) $\mathbf{x}^{(1)}(t_0), \dots, \mathbf{x}^{(m)}(t_0)$ are linearly dependent. In other words $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are linearly dependent on the interval (α, β) if they are linearly dependent at any point in it.
Hint: There are constants $c_1, \dots, c_m$ that satisfy $c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_m\mathbf{x}^{(m)}(t_0) = \mathbf{0}$. Let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t)$, and use the uniqueness theorem to show that $\mathbf{z}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$.

9. Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ be linearly independent solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, where $\mathbf{P}$ is continuous on $\alpha < t < \beta$.

form (27), provided that there are n linearly independent eigenvectors, but in general all the solutions are complex-valued.

7.5

PROBLEMS

In each of Problems 1 through 6:

(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

(b) Draw a direction field and plot a few trajectories of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}\mathbf{x}$

In each of Problems 7 and 8:

(a) Find the general solution of the given system of equations.

(b) Draw a direction field and a few of the trajectories. In each of these problems, the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text.

7. $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\mathbf{x}$

In each of Problems 9 through 14, find the general solution of the given system of equations.

9. $\mathbf{x}' = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\mathbf{x}$

10. $\mathbf{x}' = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix}\mathbf{x}$

11. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}\mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\mathbf{x}$

13. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}\mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\mathbf{x}$

In each of Problems 15 through 18, solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

15. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

16. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

17. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

18. $\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$

19. The system $t\mathbf{x}' = \mathbf{A}\mathbf{x}$ is analogous to the second order Euler equation (Section 5.4). Assuming that $\mathbf{x} = \boldsymbol{\xi} t^r$, where $\boldsymbol{\xi}$ is a constant vector, show that $\boldsymbol{\xi}$ and r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ in order to obtain nontrivial solutions of the given differential equation.

Referring to Problem 19, solve the given system of equations in each of Problems [illegible] 23. Assume that $t > 0$.

20. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}$

21. $t\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}$

22. $t\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}$

23. $t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$

In each of Problems 24 through 27, the eigenvalues and eigenvectors of a matrix $\mathbf{A}$ [illegible] Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) Sketch a phase portrait of the system.

(b) Sketch the trajectory passing through the initial point $(2, 3)$.

(c) For the trajectory in part (b), sketch the graphs of x_1 versus t and of x_2 vers[illegible] same set of axes.

24. $r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

25. $r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

26. $r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

27. $r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

28. Consider a 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If we assume that $r_1 \neq r_2$, the genera[illegible] $\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$, provided that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent. [illegible] lem we establish the linear independence of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ by assuming that they [illegible] dependent and then showing that this leads to a contradiction.

(a) Note that $\boldsymbol{\xi}^{(1)}$ satisfies the matrix equation $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}$; similar[illegible] $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(2)} = \mathbf{0}$.

(b) Show that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = (r_1 - r_2)\boldsymbol{\xi}^{(1)}$.

(c) Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$ [illegible] one of c_1 and c_2 (say c_1) is not zero. Show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = \mathbf{0}$, a[illegible] that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}$. Hence $c_1 = 0$, which is a co[illegible] Therefore, $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent.

(d) Modify the argument of part (c) if we assume that $c_2 \neq 0$.

(e) Carry out a similar argument for the case in which the order n is equal to [illegible] the procedure can be extended to an arbitrary value of n.

29. Consider the equation

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants with $a \neq 0$. In Chapter 3 it was shown that [illegible] solution depended on the roots of the characteristic equation

$$ar^2 + br + c = 0.$$

(a) Transform Eq. (i) into a system of first order equations by letting $x_1 = y$, $x_2 =$ [illegible] the system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ satisfied by $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.