7.6

PROBLEMS

In each of Problems 1 through 6:

(a) Express the general solution of the given system of equations in terms of real-valued functions.

(b) Also draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $t \to \infty$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

In each of Problems 7 and 8, express the general solution of the given system of equations in terms of real-valued functions.

7. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 9 and 10, find the solution of the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

9. $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the eigenvalues of the given system.

(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the x_1x_2-plane.

(c) For your trajectory in part (b), draw the graphs of x_1 versus t and of x_2 versus t.

(d) For your trajectory in part (b), draw the corresponding graph in three-dimensional tx_1x_2-space.

11. $\mathbf{x}' = \begin{pmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{pmatrix} \mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{pmatrix} \mathbf{x}$

In each of Problems 13 through 20, the coefficient matrix contains a parameter α. In each of these problems:

(a) Determine the eigenvalues in terms of α.

(b) Find the critical value or values of α where the qualitative nature of the phase portrait for the system changes.

(c) Draw a phase portrait for a value of α slightly below, and for another value slightly above, each critical value.

13. $\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}$

15. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}$

16. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \alpha & \frac{5}{4} \end{pmatrix} \mathbf{x}$

17. $\mathbf{x}' = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix} \mathbf{x}$

18. $\mathbf{x}' = \begin{pmatrix} 3 & \alpha \\ -6 & -4 \end{pmatrix} \mathbf{x}$

19. $\mathbf{x}' = \begin{pmatrix} \alpha & 10 \\ -1 & -4 \end{pmatrix} \mathbf{x}$

20. $\mathbf{x}' = \begin{pmatrix} 4 & \alpha \\ 8 & -6 \end{pmatrix} \mathbf{x}$

In each of Problems 21 and 22, solve the given system of equations by the meth... 19 of Section 7.5. Assume that $t > 0$.

21. $t\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x}$

22. $t\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

In each of Problems 23 and 24:

(a) Find the eigenvalues of the given system.

(b) Choose an initial point (other than the origin) and draw the correspondi... the x_1x_2-plane. Also draw the trajectories in the x_1x_3- and x_2x_3-planes.

(c) For the initial point in part (b), draw the corresponding trajectory in $x_1x_2x_3$...

23. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x}$

24. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \mathbf{x}$

25. Consider the electric circuit shown in Figure 7.6.6. Suppose that R... = $C = \frac{1}{2}$ F, and $L = 8$ H.

(a) Show that this circuit is described by the system of differential equati...

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix},$$

where I is the current through the inductor and V is the voltage drop acros... *Hint:* See Problem 20 of Section 7.1.

(b) Find the general solution of Eqs. (i) in terms of real-valued functions.

(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ A and $V(0) = 3$ V.

(d) Determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these ... depend on the initial conditions?

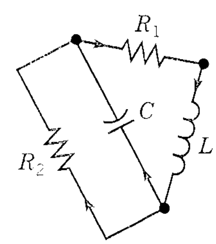

FIGURE 7.6.6 The circuit in Problem 25.

26. The electric circuit shown in Figure 7.6.7 is described by the system of differen...

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix},$$

where I is the current through the inductor and V is the voltage drop across the capacitor. These differential equations were derived in Problem 19 of Section 7.1.

(a) Show that the eigenvalues of the coefficient matrix are real and different if $L > 4R^2C$; show that they are complex conjugates if $L < 4R^2C$.

(b) Suppose that $R = 1\ \Omega$, $C = \frac{1}{2}$ F, and $L = 1$ H. Find the general solution of the system (i) in this case.

(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ A and $V(0) = 1$ V.

(d) For the circuit of part (b) determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

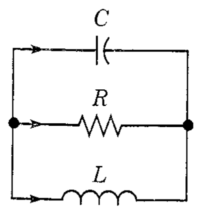

FIGURE 7.6.7 The circuit in Problem 26.

27. In this problem we indicate how to show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$, as given by Eqs. (17), are linearly independent. Let $r_1 = \lambda + i\mu$ and $\bar{r}_1 = \lambda - i\mu$ be a pair of conjugate eigenvalues of the coefficient matrix $\mathbf{A}$ of Eq. (1); let $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ and $\bar{\boldsymbol{\xi}}^{(1)} = \mathbf{a} - i\mathbf{b}$ be the corresponding eigenvectors. Recall that it was stated in Section 7.3 that two different eigenvalues have linearly independent eigenvectors, so if $r_1 \neq \bar{r}_1$, then $\boldsymbol{\xi}^{(1)}$ and $\bar{\boldsymbol{\xi}}^{(1)}$ are linearly independent.

(a) First we show that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Consider the equation $c_1\mathbf{a} + c_2\mathbf{b} = \mathbf{0}$. Express $\mathbf{a}$ and $\mathbf{b}$ in terms of $\boldsymbol{\xi}^{(1)}$ and $\bar{\boldsymbol{\xi}}^{(1)}$, and then show that $(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\bar{\boldsymbol{\xi}}^{(1)} = \mathbf{0}$.

(b) Show that $c_1 - ic_2 = 0$ and $c_1 + ic_2 = 0$ and then that $c_1 = 0$ and $c_2 = 0$. Consequently, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent.

(c) To show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent, consider the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{0}$, where t_0 is an arbitrary point. Rewrite this equation in terms of $\mathbf{a}$ and $\mathbf{b}$, and then proceed as in part (b) to show that $c_1 = 0$ and $c_2 = 0$. Hence $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent at the arbitrary point t_0. Therefore, they are linearly independent at every point and on every interval.

28. A mass m on a spring with constant k satisfies the differential equation (see Section 3.7)

$$mu'' + ku = 0,$$

where $u(t)$ is the displacement at time t of the mass from its equilibrium position.

(a) Let $x_1 = u, x_2 = u'$, and show that the resulting system is

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix}\mathbf{x}.$$

(b) Find the eigenvalues of the matrix for the system in part (a).

(c) Sketch several trajectories of the system. Choose one of your trajectories, and sketch the corresponding graphs of x_1 versus t and x_2 versus t. Sketch both graphs on one set of axes.

(d) What is the relation between the eigenvalues of the coefficient matrix and the natural frequency of the spring–mass system?

The columns of $\Psi(t)$ are the same as the solutions in Eq. (27) of Section 7.5. Thus the diagonalization procedure does not offer any computational advantage over the method of Section 7.5, since in either case it is necessary to calculate the eigenvalues and eigenvectors of the coefficient matrix in the system of differential equations.

EXAMPLE 4

Consider again the system of differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{45}$$

where $\mathbf{A}$ is given by Eq. (33). Using the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, where $\mathbf{T}$ is given by Eq. (35), you can reduce the system (45) to the diagonal system

$$\mathbf{y}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} = \mathbf{D}\mathbf{y}. \tag{46}$$

Obtain a fundamental matrix for the system (46), and then transform it to obtain a fundamental matrix for the original system (45).

By multiplying $\mathbf{D}$ repeatedly with itself, we find that

$$\mathbf{D}^2 = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D}^3 = \begin{pmatrix} 27 & 0 \\ 0 & -1 \end{pmatrix}, \quad \ldots. \tag{47}$$

Therefore, it follows from Eq. (23) that $\exp(\mathbf{D}t)$ is a diagonal matrix with the entries e^{3t} and e^{-t} on the diagonal; that is,

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{48}$$

Finally, we obtain the required fundamental matrix $\Psi(t)$ by multiplying $\mathbf{T}$ and $\exp(\mathbf{D}t)$:

$$\Psi(t) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}. \tag{49}$$

Observe that this fundamental matrix is the same as the one found in Example 1.

7.7

PROBLEMS

In each of Problems 1 through 10:

(a) Find a fundamental matrix for the given system of equations.

(b) Also find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = \mathbf{I}$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

7. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

9. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \mathbf{x}$

10. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}$

11. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

by using the fundamental matrix $\mathbf{\Phi}(t)$ found in Problem 3.

12. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

by using the fundamental matrix $\mathbf{\Phi}(t)$ found in Problem 6.

13. Show that $\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$, where $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ are as defined in this sectio

14. The fundamental matrix $\mathbf{\Phi}(t)$ for the system (3) was found in Example 2. S $\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(t+s)$ by multiplying $\mathbf{\Phi}(t)$ and $\mathbf{\Phi}(s)$.

15. Let $\mathbf{\Phi}(t)$ denote the fundamental matrix satisfying $\mathbf{\Phi}' = \mathbf{A\Phi}$, $\mathbf{\Phi}(0) = \mathbf{I}$. In the te denoted this matrix by $\exp(\mathbf{A}t)$. In this problem we show that $\mathbf{\Phi}$ does indee principal algebraic properties associated with the exponential function.

(a) Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(t+s)$; that is, show that $\exp(\mathbf{A}t)\exp(\mathbf{A}s) = \exp$ *Hint:* Show that if s is fixed and t is variable, then both $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ and $\mathbf{\Phi}(t+s$ initial value problem $\mathbf{Z}' = \mathbf{AZ}$, $\mathbf{Z}(0) = \mathbf{\Phi}(s)$.

(b) Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{I}$; that is, $\exp(\mathbf{A}t)\exp[\mathbf{A}(-t)] = \mathbf{I}$. Then show that $\mathbf{\Phi}(-t) = \mathbf{\Phi}^{-1}(t)$.

(c) Show that $\mathbf{\Phi}(t-s) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$.

16. Show that if $\mathbf{A}$ is a diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$, then also a diagonal matrix with diagonal elements $\exp(a_1 t), \exp(a_2 t), \ldots, \exp(a_n t$

17. Consider an oscillator satisfying the initial value problem

$$u'' + \omega^2 u = 0, \qquad u(0) = u_0, \quad u'(0) = v_0.$$

(a) Let $x_1 = u, x_2 = u'$, and transform Eqs. (i) into the form

$$\mathbf{x}' = \mathbf{Ax}, \qquad \mathbf{x}(0) = \mathbf{x}^0.$$

(b) By using the series (23), show that

$$\exp \mathbf{A}t = \mathbf{I}\cos\omega t + \mathbf{A}\frac{\sin\omega t}{\omega}.$$

(c) Find the solution of the initial value problem (ii).

18. The method of successive approximations (see Section 2.8) can also be applie of equations. For example, consider the initial value problem

$$\mathbf{x}' = \mathbf{Ax}, \qquad \mathbf{x}(0) = \mathbf{x}^0,$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{x}^0$ is a prescribed vector.

7.8

PROBLEMS

In each of Problems 1 through 4:

(a) Draw a direction field and sketch a few trajectories.

(b) Describe how the solutions behave as $t \to \infty$.

(c) Find the general solution of the system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix} \mathbf{x}$

In each of Problems 5 and 6, find the general solution of the given system of eq

5. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 7 through 10:

(a) Find the solution of the given initial value problem.

(b) Draw the trajectory of the solution in the x_1x_2-plane, and also draw th versus t.

7. $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

8. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

9. $\mathbf{x}' = \begin{pmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the solution of the given initial value problem.

(b) Draw the corresponding trajectory in $x_1x_2x_3$-space, and also draw the grap

11. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

12. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

In each of Problems 13 and 14, solve the given system of equations by the meth 19 of Section 7.5. Assume that $t > 0$.

13. $t\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

14. $t\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}$

15. Show that all solutions of the system

$$\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}$$

approach zero as $t \to \infty$ if and only if $a + d < 0$ and $ad - bc > 0$. Compare this result with that of Problem 37 in Section 3.4.

16. Consider again the electric circuit in Problem 26 of Section 7.6. This circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

(a) Show that the eigenvalues are real and equal if $L = 4R^2C$.
(b) Suppose that $R = 1\ \Omega$, $C = 1$ F, and $L = 4$ H. Suppose also that $I(0) = 1$ A and $V(0) = 2$ V. Find $I(t)$ and $V(t)$.

17. Consider again the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} \qquad \text{(i)}$$

that we discussed in Example 2. We found there that $\mathbf{A}$ has a double eigenvalue $r_1 = r_2 = 2$ with a single independent eigenvector $\boldsymbol{\xi}^{(1)} = (1, -1)^T$, or any nonzero multiple thereof. Thus one solution of the system (i) is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{2t}$ and a second independent solution has the form

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t},$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \qquad \text{(ii)}$$

In the text we solved the first equation for $\boldsymbol{\xi}$ and then the second equation for $\boldsymbol{\eta}$. Here we ask you to proceed in the reverse order.
(a) Show that $\boldsymbol{\eta}$ satisfies $(\mathbf{A} - 2\mathbf{I})^2\boldsymbol{\eta} = \mathbf{0}$.
(b) Show that $(\mathbf{A} - 2\mathbf{I})^2 = \mathbf{0}$. Thus the generalized eigenvector $\boldsymbol{\eta}$ can be chosen arbitrarily, except that it must be independent of $\boldsymbol{\xi}^{(1)}$.
(c) Let $\boldsymbol{\eta} = (0, -1)^T$. Then determine $\boldsymbol{\xi}$ from the second of Eqs. (ii) and observe that $\boldsymbol{\xi} = (1, -1)^T = \boldsymbol{\xi}^{(1)}$. This choice of $\boldsymbol{\eta}$ reproduces the solution found in Example 2.
(d) Let $\boldsymbol{\eta} = (1, 0)^T$ and determine the corresponding eigenvector $\boldsymbol{\xi}$.
(e) Let $\boldsymbol{\eta} = (k_1, k_2)^T$, where k_1 and k_2 are arbitrary numbers. Then determine $\boldsymbol{\xi}$. How is it related to the eigenvector $\boldsymbol{\xi}^{(1)}$?

Eigenvalues of Multiplicity 3. If the matrix $\mathbf{A}$ has an eigenvalue of algebraic multiplicity 3, then there may be either one, two, or three corresponding linearly independent eigenvectors. The general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is different, depending on the number of eigenvectors associated with the triple eigenvalue. As noted in the text, there is no difficulty if there are three eigenvectors, since then there are three independent solutions of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$. The following two problems illustrate the solution procedure for a triple eigenvalue with one or two eigenvectors, respectively.

18. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x}.$$

(a) Show that $r = 2$ is an eigenvalue of algebraic multiplicity 3 of the coefficient matrix $\mathbf{A}$ and that there is only one corresponding eigenvector, namely,

$$\xi^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b) Using the information in part (a), write down one solution $\mathbf{x}^{(1)}$ There is no other solution of the purely exponential form $\mathbf{x} = \xi e^{rt}$.

(c) To find a second solution, assume that $\mathbf{x} = \xi t e^{2t} + \eta e^{2t}$. Show that ξ equations

$$(\mathbf{A} - 2\mathbf{I})\xi = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\eta = \xi.$$

Since ξ has already been found in part (a), solve the second equation multiple of $\xi^{(1)}$ that appears in η, since it leads only to a multiple of the Then write down a second solution $\mathbf{x}^{(2)}(t)$ of the system (i).

(d) To find a third solution, assume that $\mathbf{x} = \xi(t^2/2)e^{2t} + \eta t e^{2t} + \zeta e^{2t}$. Show satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\xi = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\eta = \xi, \qquad (\mathbf{A} - 2\mathbf{I})\zeta = \eta.$$

The first two equations are the same as in part (c), so solve the third neglecting the multiple of $\xi^{(1)}$ that appears. Then write down a third system (i).

(e) Write down a fundamental matrix $\Psi(t)$ for the system (i).

(f) Form a matrix $\mathbf{T}$ with the eigenvector $\xi^{(1)}$ in the first column and eigenvectors η and ζ in the second and third columns. Then find $\mathbf{T}^{-1}$ and $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

19. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x}.$$

(a) Show that $r = 1$ is a triple eigenvalue of the coefficient matrix $\mathbf{A}$ and only two linearly independent eigenvectors, which we may take as

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad \xi^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}.$$

Write down two linearly independent solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ of Eq.

(b) To find a third solution, assume that $\mathbf{x} = \xi t e^t + \eta e^t$; then show that ξ and

$$(\mathbf{A} - \mathbf{I})\xi = \mathbf{0},$$

$$(\mathbf{A} - \mathbf{I})\eta = \xi.$$