efficiently) by using appropriate computer software. In any case, after some simplification the result is

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}. \tag{48}$$

Equation (48) gives the particular solution of the system (40) that satisfies the initial condition $\mathbf{x}(0) = \mathbf{0}$. As a result, it differs slightly from the particular solutions obtained in the preceding three examples. To obtain the general solution of Eq. (40), you must add to the expression in Eq. (48) the general solution (10) of the homogeneous system corresponding to Eq. (40).

Each of the methods for solving nonhomogeneous equations has some advantages and disadvantages. The method of undetermined coefficients requires no integration, but it is limited in scope and may entail the solution of several sets of algebraic equations. The method of diagonalization requires finding the inverse of the transformation matrix and the solution of a set of uncoupled first order linear equations, followed by a matrix multiplication. Its main advantage is that for Hermitian coefficient matrices, the inverse of the transformation matrix can be written down without calculation—a feature that is more important for large systems. The method of Laplace transforms involves a matrix inversion to find the transfer matrix, followed by a multiplication, and finally by the determination of the inverse transform of each term in the resulting expression. It is particularly useful in problems with forcing functions that involve discontinuous or impulsive terms. Variation of parameters is the most general method. On the other hand, it involves the solution of a set of linear algebraic equations with variable coefficients, followed by an integration and a matrix multiplication, so it may also be the most complicated from a computational viewpoint. For many small systems with constant coefficients, such as the one in the examples in this section, all of these methods work well, and there may be little reason to select one over another.

7.9

PROBLEMS

In each of Problems 1 through 12 find the general solution of the given system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ \sqrt{3}\, e^{-t} \end{pmatrix}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}$

5. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \quad t > 0$

6. $\mathbf{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}, \quad t > 0$

7. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$

8. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$

9. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2t \\ e^t \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

11. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}, \qquad 0 < t < \pi$

12. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}, \qquad \frac{\pi}{2} < t < \pi$

13. The electric circuit shown in Figure 7.9.1 is described by the system of differential e...

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} I(t),$$

where x_1 is the current through the inductor, x_2 is the voltage drop across the ca... and $I(t)$ is the current supplied by the external source.

(a) Determine a fundamental matrix $\boldsymbol{\Psi}(t)$ for the homogeneous system corresp... Eq. (i). Refer to Problem 25 of Section 7.6.

(b) If $I(t) = e^{-t/2}$, determine the solution of the system (i) that also satisfies th... conditions $\mathbf{x}(0) = \mathbf{0}$.

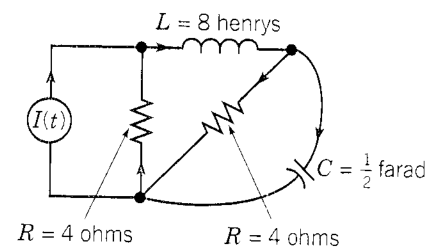

FIGURE 7.9.1 The circuit in Problem 13.

In each of Problems 14 and 15, verify that the given vector is the general solution... corresponding homogeneous system, and then solve the nonhomogeneous system. A... that $t > 0$.

14. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 - t^2 \\ 2t \end{pmatrix}, \qquad \mathbf{x}^{(c)} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}$

15. $t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2t \\ t^4 - 1 \end{pmatrix}, \qquad \mathbf{x}^{(c)} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2$

16. Let $\mathbf{x} = \boldsymbol{\phi}(t)$ be the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, and let $\mathbf{x} = \mathbf{v}(t)$ be... particular solution of the same system. By considering the difference $\boldsymbol{\phi}(t) - \mathbf{v}$... that $\boldsymbol{\phi}(t) = \mathbf{u}(t) + \mathbf{v}(t)$, where $\mathbf{u}(t)$ is the general solution of the homogeneous... $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

17. Consider the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(0) = \mathbf{x}^0.$$

(a) By referring to Problem 15(c) in Section 7.7, show that

$$\mathbf{x} = \boldsymbol{\Phi}(t)\mathbf{x}^0 + \int_0^t \boldsymbol{\Phi}(t - s)\mathbf{g}(s)\, ds.$$

6.1

PROBLEMS

In each of Problems 1 through 4, sketch the graph of the given function. In each case determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

1. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 2 + t, & 1 < t \le 2 \\ 6 - t, & 2 < t \le 3 \end{cases}$

2. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ (t-1)^{-1}, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

3. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3 - t, & 2 < t \le 3 \end{cases}$

4. $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 3 - t, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

5. Find the Laplace transform of each of the following functions:
   (a) $f(t) = t$
   (b) $f(t) = t^2$
   (c) $f(t) = t^n$, where n is a positive integer
6. Find the Laplace transform of $f(t) = \cos at$, where a is a real constant.

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10, find the Laplace transform of the given function; a and b are real constants.

7. $f(t) = \cosh bt$
8. $f(t) = \sinh bt$
9. $f(t) = e^{at} \cosh bt$
10. $f(t) = e^{at} \sinh bt$

Recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and that $\sin bt = (e^{ibt} - e^{-ibt})/2i$. In each of Problems 11 through 14, find the Laplace transform of the given function; a and b are real constants. Assume that the necessary elementary integration formulas extend to this case.

11. $f(t) = \sin bt$
12. $f(t) = \cos bt$
13. $f(t) = e^{at} \sin bt$
14. $f(t) = e^{at} \cos bt$

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

15. $f(t) = te^{at}$
16. $f(t) = t \sin at$
17. $f(t) = t \cosh at$
18. $f(t) = t^n e^{at}$
19. $f(t) = t^2 \sin at$
20. $f(t) = t^2 \sinh at$

In each of Problems 21 through 24, find the Laplace transform of the given function.

21. $f(t) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$

22. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}$

23. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < \infty \end{cases}$

24. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}$

In each of Problems 25 through 28, determine whether the given integral converges or diverges.

25. $\int_0^\infty (t^2 + 1)^{-1}\, dt$
26. $\int_0^\infty te^{-t}\, dt$
27. $\int_1^\infty t^{-2} e^t\, dt$
28. $\int_0^\infty e^{-t} \cos t\, dt$

The most important elementary applications of the Laplace transform ... study of mechanical vibrations and in the analysis of electric circuits; th... ing equations were derived in Section 3.7. A vibrating spring–mass syste... equation of motion

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = F(t),$$

where m is the mass, γ the damping coefficient, k the spring constant, an... applied external force. The equation that describes an electric circuit con... inductance L, a resistance R, and a capacitance C (an LRC circuit) is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t),$$

where $Q(t)$ is the charge on the capacitor and $E(t)$ is the applied voltage. I... the current $I(t) = dQ(t)/dt$, we can differentiate Eq. (32) and write

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}(t).$$

Suitable initial conditions on u, Q, or I must also be prescribed.

We have noted previously, in Section 3.7, that Eq. (31) for the spring–ma... and Eqs. (32) or (33) for the electric circuit are identical mathematically, ... only in the interpretation of the constants and variables appearing in the... are other physical problems that also lead to the same differential equat... once the mathematical problem is solved, its solution can be interpreted in ... whichever corresponding physical problem is of immediate interest.

In the problem lists following this and other sections in this chapter are ... initial value problems for second order linear differential equations with ... coefficients. Many can be interpreted as models of particular physical sys... usually we do not point this out explicitly.

6.2

PROBLEMS

In each of Problems 1 through 10, find the inverse Laplace transform of the given ...

1. $F(s) = \dfrac{3}{s^2+4}$
2. $F(s) = \dfrac{4}{(s-1)^3}$
3. $F(s) = \dfrac{2}{s^2+3s-4}$
4. $F(s) = \dfrac{3s}{s^2-s-6}$
5. $F(s) = \dfrac{2s+2}{s^2+2s+5}$
6. $F(s) = \dfrac{2s-3}{s^2-4}$
7. $F(s) = \dfrac{2s+1}{s^2-2s+2}$
8. $F(s) = \dfrac{8s^2-4s+12}{s(s^2+4)}$
9. $F(s) = \dfrac{1-2s}{s^2+4s+5}$
10. $F(s) = \dfrac{2s-3}{s^2+2s+10}$

In each of Problems 11 through 23, use the Laplace transform to solve the given in... problem.

11. $y'' - y' - 6y = 0; \quad y(0) = 1, \quad y'(0) = -1$
12. $y'' + 3y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 0$

13. $y'' - 2y' + 2y = 0;$ $y(0) = 0,$ $y'(0) = 1$
14. $y'' - 4y' + 4y = 0;$ $y(0) = 1,$ $y'(0) = 1$
15. $y'' - 2y' + 4y = 0;$ $y(0) = 2,$ $y'(0) = 0$
16. $y'' + 2y' + 5y = 0;$ $y(0) = 2,$ $y'(0) = -1$
17. $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0;$ $y(0) = 0,$ $y'(0) = 1,$ $y''(0) = 0,$ $y'''(0) = 1$
18. $y^{(4)} - y = 0;$ $y(0) = 1,$ $y'(0) = 0,$ $y''(0) = 1,$ $y'''(0) = 0$
19. $y^{(4)} - 4y = 0;$ $y(0) = 1,$ $y'(0) = 0,$ $y''(0) = -2,$ $y'''(0) = 0$
20. $y'' + \omega^2 y = \cos 2t,$ $\omega^2 \neq 4;$ $y(0) = 1,$ $y'(0) = 0$
21. $y'' - 2y' + 2y = \cos t;$ $y(0) = 1,$ $y'(0) = 0$
22. $y'' - 2y' + 2y = e^{-t};$ $y(0) = 0,$ $y'(0) = 1$
23. $y'' + 2y' + y = 4e^{-t};$ $y(0) = 2,$ $y'(0) = -1$

In each of Problems 24 through 27, find the Laplace transform $Y(s) = \mathcal{L}\{y\}$ of the solution of the given initial value problem. A method of determining the inverse transform is developed in Section 6.3. You may wish to refer to Problems 21 through 24 in Section 6.1.

24. $y'' + 4y = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < \infty; \end{cases}$ $y(0) = 1,$ $y'(0) = 0$

25. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < \infty; \end{cases}$ $y(0) = 0,$ $y'(0) = 0$

26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ $y(0) = 0,$ $y'(0) = 0$

27. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2, \\ 0, & 2 \le t < \infty; \end{cases}$ $y(0) = 0,$ $y'(0) = 0$

28. The Laplace transforms of certain functions can be found conveniently from their Taylor series expansions.
(a) Using the Taylor series for $\sin t$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!},$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \qquad s > 1.$$

(b) Let

$$f(t) = \begin{cases} (\sin t)/t, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that

$$\mathcal{L}\{f(t)\} = \arctan(1/s), \qquad s > 1.$$