As usual, the eigenfunctions $y_n(x)$ are determined only up to an arbitrary multiplicative constant. In the same way as for the problem (18), (19), you can show that the problem (29) has no eigenvalues or eigenfunctions other than those in Eq. (31).

The problems following this section explore to some extent the effect of different boundary conditions on the eigenvalues and eigenfunctions. A more systematic discussion of two-point boundary and eigenvalue problems appears in Chapter 11.

10.1

PROBLEMS

In each of Problems 1 through 13, either solve the given boundary value problem or else show that it has no solution.

1. $y'' + y = 0, \quad y(0) = 0, \quad y'(\pi) = 1$
2. $y'' + 2y = 0, \quad y'(0) = 1, \quad y'(\pi) = 0$
3. $y'' + y = 0, \quad y(0) = 0, \quad y(L) = 0$
4. $y'' + y = 0, \quad y'(0) = 1, \quad y(L) = 0$
5. $y'' + y = x, \quad y(0) = 0, \quad y(\pi) = 0$
6. $y'' + 2y = x, \quad y(0) = 0, \quad y(\pi) = 0$
7. $y'' + 4y = \cos x, \quad y(0) = 0, \quad y(\pi) = 0$
8. $y'' + 4y = \sin x, \quad y(0) = 0, \quad y(\pi) = 0$
9. $y'' + 4y = \cos x, \quad y'(0) = 0, \quad y'(\pi) = 0$
10. $y'' + 3y = \cos x, \quad y'(0) = 0, \quad y'(\pi) = 0$
11. $x^2y'' - 2xy' + 2y = 0, \quad y(1) = -1, \quad y(2) = 1$
12. $x^2y'' + 3xy' + y = x^2, \quad y(1) = 0, \quad y(e) = 0$
13. $x^2y'' + 5xy' + (4 + \pi^2)y = \ln x, \quad y(1) = 0, \quad y(e) = 0$

In each of Problems 14 through 20, find the eigenvalues and eigenfunctions of the given boundary value problem. Assume that all eigenvalues are real.

14. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$
15. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0$
16. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$
17. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$
18. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$
19. $y'' - \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$
20. $x^2y'' - xy' + \lambda y = 0, \quad y(1) = 0, \quad y(L) = 0, \quad L > 1$
21. The axially symmetric laminar flow of a viscous incompressible fluid through a long straight tube of circular cross section under a constant axial pressure gradient is known as Poiseuille[1] flow. The axial velocity w is a function of the radial variable r only and satisfies the boundary value problem

$$w'' + \frac{1}{r}w' = -\frac{G}{\mu}, \qquad w(R) = 0, \qquad w(r) \text{ bounded for } 0 < r < R,$$

where R is the radius of the tube, G is the pressure gradient, and μ is the coefficient of viscosity of the fluid.

(a) Find the velocity profile $w(r)$.

(b) By integrating $w(r)$ over a cross section, show that the total flow rate Q is given by

$$Q = \pi R^4 G/8\mu.$$

Since Q, R, and G can be measured, this result provides a practical way to determine the viscosity μ.

(c) Suppose that R is reduced to 3/4 of its original value. What is the corresponding reduction in Q? This result has implications for blood flow through arteries constricted by plaque.

[1] Jean Louis Marie Poiseuille (1797–1869) was a French physician who was also trained in mathematics and physics. He was particularly interested in the flow of blood and published his first paper on the subject in 1840.

10.2

PROBLEMS

In each of Problems 1 through 8, determine whether the given function is periodic. If so, find its fundamental period.

1. $\sin 5x$ 2. $\cos 2\pi x$ 3. $\sinh 2x$ 4. $\sin \pi x/L$ 5. $\tan \pi x$ 6. x^2

7. $f(x) = \begin{cases} 0, & 2n-1 \le x < 2n, \\ 1, & 2n \le x < 2n+1; \end{cases} \quad n = 0, \pm 1, \pm 2, \ldots$

8. $f(x) = \begin{cases} (-1)^n, & 2n-1 \le x < 2n, \\ 1, & 2n \le x < 2n+1; \end{cases} \quad n = 0, \pm 1, \pm 2, \ldots$

9. If $f(x) = -x$ for $-L < x < L$, and if $f(x+2L) = f(x)$, find a formula for $f(x)$ in the interval $L < x < 2L$; in the interval $-3L < x < -2L$.

10. If $f(x) = \begin{cases} x+1, & -1 < x < 0, \\ x, & 0 < x < 1, \end{cases}$ and if $f(x+2) = f(x)$, find a formula for $f(x)$ in the interval $1 < x < 2$; in the interval $8 < x < 9$.

11. If $f(x) = L - x$ for $0 < x < 2L$, and if $f(x+2L) = f(x)$, find a formula for $f(x)$ in the interval $-L < x < 0$.

12. Verify Eqs. (6) and (7) in this section by direct integration.

In each of Problems 13 through 18:

(a) Sketch the graph of the given function for three periods.

(b) Find the Fourier series for the given function.

13. $f(x) = -x, \quad -L \le x < L; \qquad f(x+2L) = f(x)$

14. $f(x) = \begin{cases} 1, & -L \le x < 0, \\ 0, & 0 \le x < L; \end{cases} \qquad f(x+2L) = f(x)$

15. $f(x) = \begin{cases} x, & -\pi \le x < 0, \\ 0, & 0 \le x < \pi; \end{cases} \qquad f(x+2\pi) = f(x)$

16. $f(x) = \begin{cases} x+1, & -1 \le x < 0, \\ 1-x, & 0 \le x < 1; \end{cases} \qquad f(x+2) = f(x)$

17. $f(x) = \begin{cases} x+L, & -L \le x \le 0, \\ L, & 0 < x < L; \end{cases} \qquad f(x+2L) = f(x)$

18. $f(x) = \begin{cases} 0, & -2 \le x \le -1, \\ x, & -1 < x < 1, \\ 0, & 1 \le x < 2; \end{cases} \qquad f(x+4) = f(x)$

In each of Problems 19 through 24:

(a) Sketch the graph of the given function for three periods.

(b) Find the Fourier series for the given function.

(c) Plot the partial sum $s_m(x)$ versus x for $m = 5, 10,$ and 20.

(d) Describe how the Fourier series seems to be converging.

19. $f(x) = \begin{cases} -1, & -2 \le x < 0, \\ 1, & 0 \le x < 2; \end{cases} \qquad f(x+4) = f(x)$

20. $f(x) = x, \quad -1 \le x < 1; \qquad f(x+2) = f(x)$

10.3

PROBLEMS

In each of Problems 1 through 6, assume that the given function is periodic… outside the original interval.

(a) Find the Fourier series for the extended function.

(b) Sketch the graph of the function to which the series converges for three pe…

1. $f(x) = \begin{cases} -1, & -1 \le x < 0, \\ 1, & 0 \le x < 1 \end{cases}$

2. $f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ x, & 0 \le x < \pi \end{cases}$

3. $f(x) = \begin{cases} L + x, & -L \le x < 0, \\ L - x, & 0 \le x < L \end{cases}$

4. $f(x) = 1 - x^2, \quad -1 \le x <$ …

5. $f(x) = \begin{cases} 0, & -\pi \le x < -\pi/2, \\ 1, & -\pi/2 \le x < \pi/2, \\ 0, & \pi/2 \le x < \pi \end{cases}$

6. $f(x) = \begin{cases} 0, & -1 \le x < 0, \\ x^2, & 0 \le x < 1 \end{cases}$

In each of Problems 7 through 12, assume that the given function is periodic… outside the original interval.

(a) Find the Fourier series for the given function.

(b) Let $e_n(x) = f(x) - s_n(x)$. Find the least upper bound or the maximum value … of $|e_n(x)|$ for $n = 10, 20$, and 40.

(c) If possible, find the smallest n for which $|e_n(x)| \le 0.01$ for all x.

7. $f(x) = \begin{cases} x, & -\pi \le x < 0, \\ 0, & 0 \le x < \pi; \end{cases} \qquad f(x + 2\pi) = f(x)$ (see Section 10.2, Prob…

8. $f(x) = \begin{cases} x + 1, & -1 \le x < 0, \\ 1 - x, & 0 \le x < 1; \end{cases} \qquad f(x + 2) = f(x)$ (see Section 10.2, Pro…

9. $f(x) = x, \quad -1 \le x < 1; \qquad f(x + 2) = f(x)$ (see Section 10.2, Problem …

10. $f(x) = \begin{cases} x + 2, & -2 \le x < 0, \\ 2 - 2x, & 0 \le x < 2; \end{cases} \qquad f(x + 4) = f(x)$ (see Section 10.2, Pro…

11. $f(x) = \begin{cases} 0, & -1 \le x < 0, \\ x^2, & 0 \le x < 1; \end{cases} \qquad f(x + 2) = f(x)$ (see Problem 6)

12. $f(x) = x - x^3, \quad -1 \le x < 1; \qquad f(x + 2) = f(x)$

Periodic Forcing Terms. In this chapter we are concerned mainly with the use of F… to solve boundary value problems for certain partial differential equations. Howe… series are also useful in many other situations where periodic phenomena occur. P… through 16 indicate how they can be employed to solve initial value problems w… forcing terms.

13. Find the solution of the initial value problem

$$y'' + \omega^2 y = \sin nt, \qquad y(0) = 0, \quad y'(0) = 0,$$

where n is a positive integer and $\omega^2 \ne n^2$. What happens if $\omega^2 = n^2$?

14. Find the formal solution of the initial value problem

$$y'' + \omega^2 y = \sum_{n=1}^{\infty} b_n \sin nt, \qquad y(0) = 0, \quad y'(0) = 0.$$

10.4

PROBLEMS

In each of Problems 1 through 6, determine whether the given function is even, ...

1. $x^3 - 2x$
2. $x^3 - 2x + 1$
3. $\tan 2x$
4. $\sec x$
5. $|x|^3$
6. e^{-x}

In each of Problems 7 through 12, a function f is given on an interval of length ... sketch the graphs of the even and odd extensions of f of period $2L$.

7. $f(x) = \begin{cases} x, & 0 \le x < 2, \\ 1, & 2 \le x < 3 \end{cases}$

8. $f(x) = \begin{cases} 0, & 0 \le x < \\ x - 1, & 1 \le x < \end{cases}$

9. $f(x) = 2 - x, \quad 0 < x < 2$

10. $f(x) = x - 3, \quad 0 < x < 4$

11. $f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & 1 \le x < 2 \end{cases}$

12. $f(x) = 4 - x^2, \quad 0 < x <$

13. Prove that any function can be expressed as the sum of two other functions ... is even and the other odd. That is, for any function f, whose domain contains ... it contains x, show that there are an even function g and an odd function ... $f(x) = g(x) + h(x)$.
Hint: What can you say about $f(x) + f(-x)$?

14. Find the coefficients in the cosine and sine series described in Example 2.

In each of Problems 15 through 22:

(a) Find the required Fourier series for the given function.
(b) Sketch the graph of the function to which the series converges over three pe...

15. $f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 2; \end{cases}$ cosine series, period 4
Compare with Example 1 and Problem 5 of Section 10.3.

16. $f(x) = \begin{cases} x, & 0 \le x < 1, \\ 1, & 1 \le x < 2; \end{cases}$ sine series, period 4

17. $f(x) = 1, \quad 0 \le x \le \pi;$ cosine series, period 2π

18. $f(x) = 1, \quad 0 < x < \pi;$ sine series, period 2π

19. $f(x) = \begin{cases} 0, & 0 < x < \pi, \\ 1, & \pi < x < 2\pi, \\ 2, & 2\pi < x < 3\pi; \end{cases}$ sine series, period 6π

20. $f(x) = x, \quad 0 \le x < 1;$ series of period 1

21. $f(x) = L - x, \quad 0 \le x \le L;$ cosine series, period $2L$
Compare with Example 1 of Section 10.2.

22. $f(x) = L - x, \quad 0 < x < L;$ sine series, period $2L$

In each of Problems 23 through 26:

(a) Find the required Fourier series for the given function.
(b) Sketch the graph of the function to which the series converges for three periods.
(c) Plot one or more partial sums of the series.

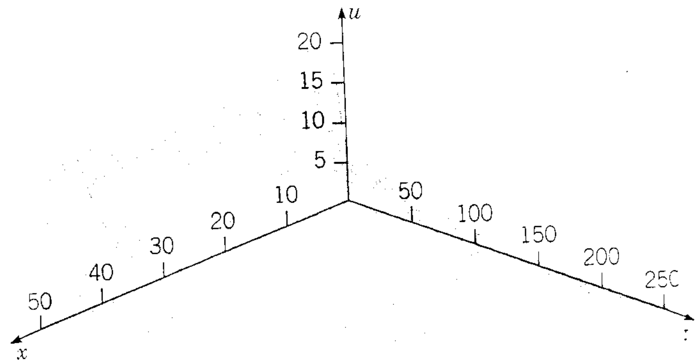

FIGURE 10.5.5 Plot of temperature u versus x and t for the heat conduction problem of Example 1.

10.5

PROBLEMS

In each of Problems 1 through 6, determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

1. $xu_{xx} + u_t = 0$
2. $tu_{xx} + xu_t = 0$
3. $u_{xx} + u_{xt} + u_t = 0$
4. $[p(x)u_x]_x - r(x)u_{tt} = 0$
5. $u_{xx} + (x+y)u_{yy} = 0$
6. $u_{xx} + u_{yy} + xu = 0$
7. Find the solution of the heat conduction problem

$$\begin{aligned} 100u_{xx} &= u_t, \qquad 0 < x < 1, \quad t > 0; \\ u(0,t) = 0, &\qquad u(1,t) = 0, \quad t > 0; \\ u(x,0) &= \sin 2\pi x - \sin 5\pi x, \qquad 0 \le x \le 1. \end{aligned}$$

8. Find the solution of the heat conduction problem

$$\begin{aligned} u_{xx} &= 4u_t, \qquad 0 < x < 2, \quad t > 0; \\ u(0,t) = 0, &\qquad u(2,t) = 0, \quad t > 0; \\ u(x,0) &= 2\sin(\pi x/2) - \sin \pi x + 4\sin 2\pi x, \qquad 0 \le x \le 2. \end{aligned}$$

Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all $t > 0$. In each of Problems 9 through 12, find an expression for the temperature $u(x,t)$ if the initial temperature distribution in the rod is the given function. Suppose that $\alpha^2 = 1$.

9. $u(x,0) = 50, \qquad 0 < x < 40$

10. $u(x,0) = \begin{cases} x, & 0 \le x < 20, \\ 40 - x, & 20 \le x \le 40 \end{cases}$

reduces to

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial u}{\partial \tau}, \qquad 0 < \xi < 1, \quad \tau > 0.$$

21. Consider the equation

$$au_{xx} - bu_t + cu = 0,$$

where a, b, and c are constants.

(a) Let $u(x,t) = e^{\delta t} w(x,t)$, where δ is constant, and find the corresp differential equation for w.

(b) If $b \neq 0$, show that δ can be chosen so that the partial differential eq part (a) has no term in w. Thus, by a change of dependent variable, it is pos Eq. (i) to the heat conduction equation.

22. The heat conduction equation in two space dimensions is

$$\alpha^2 (u_{xx} + u_{yy}) = u_t.$$

Assuming that $u(x,y,t) = X(x)Y(y)T(t)$, find ordinary differential equ satisfied by $X(x)$, $Y(y)$, and $T(t)$.

23. The heat conduction equation in two space dimensions may be expressed i coordinates as

$$\alpha^2 [u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta}] = u_t.$$

Assuming that $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equa satisfied by $R(r)$, $\Theta(\theta)$, and $T(t)$.

10.6 Other Heat Conduction Problems

In Section 10.5 we considered the problem consisting of the heat conducti

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad t > 0,$$

the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \quad t > 0.$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le L.$$

We found the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin \frac{n\pi x}{L},$$

where the coefficients c_n are the same as in the series

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}.$$