More General Problems. The method of separation of variables can also be used to solve heat conduction problems with boundary conditions other than those given by Eqs. (8) and Eqs. (24). For example, the left end of the bar might be held at a fixed temperature T while the other end is insulated. In this case the boundary conditions are

$$u(0,t) = T, \qquad u_x(L,t) = 0, \quad t > 0. \tag{43}$$

The first step in solving this problem is to reduce the given boundary conditions to homogeneous ones by subtracting the steady state solution. The resulting problem is solved by essentially the same procedure as in the problems previously considered. However, the extension of the initial function f outside of the interval $[0, L]$ is somewhat different from that in any case considered so far (see Problem 15).

A more general type of boundary condition occurs when the rate of heat flow through the end of the bar is proportional to the temperature. It is shown in Appendix A that the boundary conditions in this case are of the form

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad u_x(L,t) + h_2 u(L,t) = 0, \quad t > 0, \tag{44}$$

where h_1 and h_2 are nonnegative constants. If we apply the method of separation of variables to the problem consisting of Eqs. (1), (3), and (44), we find that $X(x)$ must be a solution of

$$X'' + \lambda X = 0, \qquad X'(0) - h_1 X(0) = 0, \qquad X'(L) + h_2 X(L) = 0, \tag{45}$$

where λ is the separation constant. Once again it is possible to show that nontrivial solutions can exist only for certain nonnegative real values of λ, the eigenvalues, but these values are not given by a simple formula (see Problem 20). It is also possible to show that the corresponding solutions of Eqs. (45), the eigenfunctions, satisfy an orthogonality relation and that we can satisfy the initial condition (3) by superposing solutions of Eqs. (45). However, the resulting series is not included in the discussion of this chapter. There is a more general theory that covers such problems, and it is outlined in Chapter 11.

PROBLEMS

In each of Problems 1 through 8, find the steady state solution of the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the given set of boundary conditions.

1. $u(0,t) = 10, \quad u(50,t) = 40$
2. $u(0,t) = 30, \quad u(40,t) = -20$
3. $u_x(0,t) = 0, \quad u(L,t) = 0$
4. $u_x(0,t) = 0, \quad u(L,t) = T$
5. $u(0,t) = 0, \quad u_x(L,t) = 0$
6. $u(0,t) = T, \quad u_x(L,t) = 0$
7. $u_x(0,t) - u(0,t) = 0, \quad u(L,t) = T$
8. $u(0,t) = T, \quad u_x(L,t) + u(L,t) = 0$
9. Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25°C. Suppose that at time $t = 0$, the end $x = 0$ is cooled to 0°C while the end $x = 20$ is heated to 60°C, and both are thereafter maintained at those temperatures.

   (a) Find the temperature distribution in the rod at any time t.

(b) Plot the initial temperature distribution, the final (steady state) tempera... tion, and the temperature distributions at two representative intermediate ... same set of axes.

(c) Plot u versus t for $x = 5, 10$, and 15.

(d) Determine how much time must elapse before the temperature at $x = 5$ ... (and remains) within 1% of its steady state value.

10\. (a) Let the ends of a copper rod 100 cm long be maintained at 0°C. Supp... center of the bar is heated to 100°C by an external heat source and that th... maintained until a steady state results. Find this steady state temperature dis...

(b) At a time $t = 0$ [after the steady state of part (a) has been reached], let the ... be removed. At the same instant let the end $x = 0$ be placed in thermal co... reservoir at 20°C, while the other end remains at 0°C. Find the temperature ... of position and time.

(c) Plot u versus x for several values of t. Also plot u versus t for several val...

(d) What limiting value does the temperature at the center of the rod appr... long time? How much time must elapse before the center of the rod cools ... of its limiting value?

11\. Consider a rod of length 30 for which $\alpha^2 = 1$. Suppose the initial temperatur... is given by $u(x, 0) = x(60 - x)/30$ and that the boundary conditions are $u(\ldots) = 30$, $u(30, t) = 0$.

(a) Find the temperature in the rod as a function of position and time.

(b) Plot u versus x for several values of t. Also plot u versus t for several val...

(c) Plot u versus t for $x = 12$. Observe that u initially decreases, then incr... while, and finally decreases to approach its steady state value. Explain physic... behavior occurs at this point.

12\. Consider a uniform rod of length L with an initial temperature given by $u(x, 0) = \sin(\pi x/L)$, $0 \le x \le L$. Assume that both ends of the bar are insulat...

(a) Find the temperature $u(x, t)$.

(b) What is the steady state temperature as $t \to \infty$?

(c) Let $\alpha^2 = 1$ and $L = 40$. Plot u versus x for several values of t. Also plot ... several values of x.

(d) Describe briefly how the temperature in the rod changes as time progre...

13\. Consider a bar of length 40 cm whose initial temperature is given by $u(x, 0) = \ldots$ Suppose that $\alpha^2 = 1/4$ cm^2/s and that both ends of the bar are insulated.

(a) Find the temperature $u(x, t)$.

(b) Plot u versus x for several values of t. Also plot u versus t for several val...

(c) Determine the steady state temperature in the bar.

(d) Determine how much time must elapse before the temperature at $x = 40$ ... 1°C of its steady state value.

14\. Consider a bar 30 cm long that is made of a material for which $\alpha^2 = 1$ and wh... insulated. Suppose that the initial temperature is zero except for the interva... where the initial temperature is 25°C.

(a) Find the temperature $u(x, t)$.

(b) Plot u versus x for several values of t. Also plot u versus t for several val...

(c) Plot $u(4, t)$ and $u(11, t)$ versus t. Observe that the points $x = 4$ and $x = 11$ ... rically located with respect to the initial temperature pulse, yet their temper... are significantly different. Explain physically why this is so.

15. Consider a uniform bar of length L having an initial temperature distribution given by $f(x)$, $0 \leq x \leq L$. Assume that the temperature at the end $x = 0$ is held at 0°C, while the end $x = L$ is insulated so that no heat passes through it.

(a) Show that the fundamental solutions of the partial differential equation and boundary conditions are

$$u_n(x,t) = e^{-(2n-1)^2\pi^2\alpha^2 t/4L^2}\sin[(2n-1)\pi x/2L], \qquad n = 1,2,3,\ldots.$$

(b) Find a formal series expansion for the temperature $u(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

that also satisfies the initial condition $u(x,0) = f(x)$.
Hint: Even though the fundamental solutions involve only the odd sines, it is still possible to represent f by a Fourier series involving only these functions. See Problem 39 of Section 10.4.

16. In the bar of Problem 15, suppose that $L = 30$, $\alpha^2 = 1$, and the initial temperature distribution is $f(x) = 30 - x$ for $0 < x < 30$.

(a) Find the temperature $u(x,t)$.

(b) Plot u versus x for several values of t. Also plot u versus t for several values of x.

(c) How does the location x_m of the warmest point in the bar change as t increases? Draw a graph of x_m versus t.

(d) Plot the maximum temperature in the bar versus t.

17. Suppose that the conditions are as in Problems 15 and 16 except that the boundary condition at $x = 0$ is $u(0,t) = 40$.

(a) Find the temperature $u(x,t)$.

(b) Plot u versus x for several values of t. Also plot u versus t for several values of x.

(c) Compare the plots you obtained in this problem with those from Problem 16. Explain how the change in the boundary condition at $x = 0$ causes the observed differences in the behavior of the temperature in the bar.

18. Consider the problem

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \qquad X'(L) = 0. \tag{i}$$

Let $\lambda = \mu^2$, where $\mu = \nu + i\sigma$ with ν and σ real. Show that if $\sigma \neq 0$, then the only solution of Eqs. (i) is the trivial solution $X(x) = 0$.
Hint: Use an argument similar to that in Problem 23 of Section 10.1.

19. The right end of a bar of length a with thermal conductivity κ_1 and cross-sectional area A_1 is joined to the left end of a bar of thermal conductivity κ_2 and cross-sectional area A_2. The composite bar has a total length L. Suppose that the end $x = 0$ is held at temperature zero, while the end $x = L$ is held at temperature T. Find the steady state temperature in the composite bar, assuming that the temperature and rate of heat flow are continuous at $x = a$.
Hint: See Eq. (2) of Appendix A.

20. Consider the problem

$$\begin{aligned} \alpha^2 u_{xx} &= u_t, && 0 < x < L, \quad t > 0; \\ u(0,t) = 0, \qquad u_x(L,t) + \gamma u(L,t) &= 0, && t > 0; \\ u(x,0) &= f(x), && 0 \leq x \leq L. \end{aligned} \tag{i}$$

(a) Let $u(x,t) = X(x)T(t)$, and show that

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X'(L) + \gamma X(L) = 0.$$

and

$$T' + \lambda\alpha^2 T = 0,$$

where λ is the separation constant.

(b) Assume that λ is real, and show that problem (ii) has no nontrivial so...

(c) If $\lambda > 0$, let $\lambda = \mu^2$ with $\mu > 0$. Show that problem (ii) has nontrivial s... μ is a solution of the equation

$$\mu \cos \mu L + \gamma \sin \mu L = 0.$$

(d) Rewrite Eq. (iii) as $\tan \mu L = -\mu/\gamma$. Then, by drawing the graphs of ... = ... $y = -\mu L/\gamma L$ for $\mu > 0$ on the same set of axes, show that Eq. (iii) is satisfi... many positive values of μ; denote these by $\mu_1, \mu_2, \ldots, \mu_n, \ldots$, ordered in ...

(e) Determine the set of fundamental solutions $u_n(x,t)$ corresponding to ... found in part (d).

An External Heat Source. Consider the heat conduction problem in a bar th... contact with an external heat source or sink. Then the modified heat conducti...

$$u_t = \alpha^2 u_{xx} + s(x),$$

where the term $s(x)$ describes the effect of the external agency; $s(x)$ is positive f... negative for a sink. Suppose that the boundary conditions are

$$u(0,t) = T_1, \qquad u(L,t) = T_2$$

and the initial condition is

$$u(x,0) = f(x).$$

Problems 21 through 23 deal with this kind of problem.

21. Write $u(x,t) = v(x) + w(x,t)$, where v and w are the steady state and transie... solution, respectively. State the boundary value problems that $v(x)$ and $w(x,t)$ ... satisfy. Observe that the problem for w is the fundamental heat conduc... discussed in Section 10.5, with a modified initial temperature distribution.

22. (a) Suppose that $\alpha^2 = 1$ and $s(x) = k$, a constant, in Eq. (i). Find $v(x)$.
(b) Assume that $T_1 = 0$, $T_2 = 0$, $L = 20$, $k = 1/5$, and that $f(x) = 0$ for $0 < x$ ... mine $w(x,t)$. Then plot $u(x,t)$ versus x for several values of t; on the same ax... the steady state part of the solution $v(x)$.

23. (a) Let $\alpha^2 = 1$ and $s(x) = kx/L$, where k is a constant, in Eq. (i). Find $v(x)$.
(b) Assume that $T_1 = 10$, $T_2 = 30$, $L = 20$, $k = 1/2$, and that $f(x) = 0$ fo... Determine $w(x,t)$. Then plot $u(x,t)$ versus x for several values of t; on the sa... also plot the steady state part of the solution $v(x)$.

General Problem for the Elastic String. Finally, we turn to the problem … wave equation (1), the boundary conditions (3), and the general in… (4), (5):

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 < x < L.$$

where $f(x)$ and $g(x)$ are the given initial position and velocity, resp… string. Although this problem can be solved by separating variables … discussed previously, it is important to note that it can also be solved s… together the two solutions that we obtained above. To show that this … be the solution of the problem (1), (3), and (9), and let $w(x,t)$ be the … problem (1), (3), and (31). Thus $v(x,t)$ is given by Eqs. (20) and (2… given by Eqs. (34) and (36). Now let $u(x,t) = v(x,t) + w(x,t)$; what … $u(x,t)$ satisfy? First, observe that

$$a^2 u_{xx} - u_{tt} = (a^2 v_{xx} - v_{tt}) + (a^2 w_{xx} - w_{tt}) = 0 + 0 = 0$$

so $u(x,t)$ satisfies the wave equation (1). Next, we have

$$u(0,t) = v(0,t) + w(0,t) = 0 + 0 = 0, \qquad u(L,t) = v(L,t) + w(L, \ldots$$

so $u(x,t)$ also satisfies the boundary conditions (3). Finally, we have

$$u(x,0) = v(x,0) + w(x,0) = f(x) + 0 = f(x)$$

and

$$u_t(x,0) = v_t(x,0) + w_t(x,0) = 0 + g(x) = g(x).$$

Thus $u(x,t)$ satisfies the general initial conditions (37).

We can restate the result we have just obtained in the following wa… wave equation with the general initial conditions (37), you can solve ins… what simpler problems with the initial conditions (9) and (31), respec… add together the two solutions. This is another use of the principle of …

10.7

PROBLEMS

Consider an elastic string of length L whose ends are held fixed. The string … with no initial velocity from an initial position $u(x,0) = f(x)$. In each of Prob… carry out the following steps. Let $L = 10$ and $a = 1$ in parts (b) through (d).

(a) Find the displacement $u(x,t)$ for the given initial position $f(x)$.

(b) Plot $u(x,t)$ versus x for $0 \le x \le 10$ and for several values of t between $t =$ …

(c) Plot $u(x,t)$ versus t for $0 \le t \le 20$ and for several values of x.

(d) Construct an animation of the solution in time for at least one period.

(e) Describe the motion of the string in a few sentences.

1. $f(x) = \begin{cases} 2x/L, & 0 \le x \le L/2, \\ 2(L-x)/L, & L/2 < x \le L \end{cases}$

2. $f(x) = \begin{cases} 4x/L, & 0 \le x \le L/4, \\ 1, & L/4 < x < 3L/4, \\ 4(L-x)/L, & 3L/4 \le x \le L \end{cases}$

3. $f(x) = 8x(L-x)^2/L^3$

4. $f(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \quad (L > 2), \\ 0, & \text{otherwise} \end{cases}$

Consider an elastic string of length L whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity $u_t(x, 0) = g(x)$. In each of Problems 5 through 8, carry out the following steps. Let $L = 10$ and $a = 1$ in parts (b) through (d).

(a) Find the displacement $u(x, t)$ for the given $g(x)$.

(b) Plot $u(x, t)$ versus x for $0 \le x \le 10$ and for several values of t between $t = 0$ and $t = 20$.

(c) Plot $u(x, t)$ versus t for $0 \le t \le 20$ and for several values of x.

(d) Construct an animation of the solution in time for at least one period.

(e) Describe the motion of the string in a few sentences.

5. $g(x) = \begin{cases} 2x/L, & 0 \le x \le L/2, \\ 2(L - x)/L, & L/2 < x \le L \end{cases}$

6. $g(x) = \begin{cases} 4x/L, & 0 \le x \le L/4, \\ 1, & L/4 < x < 3L/4, \\ 4(L - x)/L, & 3L/4 \le x \le L \end{cases}$

7. $g(x) = 8x(L - x)^2/L^3$

8. $g(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \quad (L > 2), \\ 0, & \text{otherwise} \end{cases}$

9. If an elastic string is free at one end, the boundary condition to be satisfied there is that $u_x = 0$. Find the displacement $u(x, t)$ in an elastic string of length L, fixed at $x = 0$ and free at $x = L$, set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where f is a given function.
Hint: Show that the fundamental solutions for this problem, satisfying all conditions except the nonhomogeneous initial condition, are

$$u_n(x, t) = \sin \lambda_n x \cos \lambda_n at,$$

where $\lambda_n = (2n - 1)\pi/(2L), n = 1, 2, \ldots$. Compare this problem with Problem 15 of Section 10.6; pay particular attention to the extension of the initial data out of the original interval $[0, L]$.

10. Consider an elastic string of length L. The end $x = 0$ is held fixed, while the end $x = L$ is free; thus the boundary conditions are $u(0, t) = 0$ and $u_x(L, t) = 0$. The string is set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where

$$f(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \quad (L > 2), \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the displacement $u(x, t)$.

(b) With $L = 10$ and $a = 1$, plot u versus x for $0 \le x \le 10$ and for several values of t. Pay particular attention to values of t between 3 and 7. Observe how the initial disturbance is reflected at each end of the string.

(c) With $L = 10$ and $a = 1$, plot u versus t for several values of x.

(d) Construct an animation of the solution in time for at least one period.

(e) Describe the motion of the string in a few sentences.

11. Suppose that the string in Problem 10 is started instead from the initial position $f(x) = 8x(L - x)^2/L^3$. Follow the instructions in Problem 10 for this new problem.

(a) Using the form of the solution obtained in Problem 13, show that

$$\phi(x) + \psi(x) = 0,$$
$$-a\phi'(x) + a\psi'(x) = g(x).$$

(b) Use the first equation of part (a) to show that $\psi'(x) = -\phi'(x)$. Then use ... equation to show that $-2a\phi'(x) = g(x)$ and therefore that

$$\phi(x) = -\frac{1}{2a}\int_{x_0}^{x} g(\xi)\,d\xi + \phi(x_0),$$

where x_0 is arbitrary. Finally, determine $\psi(x)$.

(c) Show that

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,d\xi.$$

18. By combining the results of Problems 16 and 17, show that the solution of the ...

$$a^2 u_{xx} = u_{tt},$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad -\infty < x < \infty$$

is given by

$$u(x,t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,d\xi.$$

Problems 19 and 20 indicate how the formal solution (20), (22) of Eqs. (1), (3), and ... shown to constitute the actual solution of that problem.

19. By using the trigonometric identity $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$ ... the solution (20) of the problem of Eqs. (1), (3), and (9) can be written in the ...

20. Let $h(\xi)$ represent the initial displacement in $[0, L]$, extended into $(-L, 0)$ as an ... tion and extended elsewhere as a periodic function of period $2L$. Assuming that ... h'' are continuous, show by direct differentiation that $u(x,t)$ as given in Eq. (28) sat... wave equation (1) and also the initial conditions (9). Note also that since Eq. ... satisfies the boundary conditions (3), the same is true of Eq. (28). Comparing ... with the solution of the corresponding problem for the infinite string (Problem ... that they have the same form, provided that the initial data for the finite string ... originally only on the interval $0 \le x \le L$, are extended in the given manner over ... x-axis. If this is done, the solution for the infinite string is also applicable to the ...

21. The motion of a circular elastic membrane, such as a drumhead, is governed by ... dimensional wave equation in polar coordinates

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} = a^{-2}u_{tt}.$$

Assuming that $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equations sat... $R(r)$, $\Theta(\theta)$, and $T(t)$.

22. The total energy $E(t)$ of the vibrating string is given as a function of time by

$$E(t) = \int_0^L \left[\tfrac{1}{2}\rho u_t^2(x,t) + \tfrac{1}{2}Tu_x^2(x,t)\right] dx;$$

The boundary condition (18) then requires that

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos n\theta + k_n \sin n\theta) = f(\theta) \tag{35}$$

for $0 \le \theta < 2\pi$. The function f may be extended outside this interval so that it is periodic with period 2π and therefore has a Fourier series of the form (35). Since the extended function has period 2π, we may compute its Fourier coefficients by integrating over any period of the function. In particular, it is convenient to use the original interval $(0, 2\pi)$; then

$$a^n c_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \qquad n = 0, 1, 2, \ldots; \tag{36}$$

$$a^n k_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta, \qquad n = 1, 2, \ldots. \tag{37}$$

With this choice of the coefficients, Eq. (34) represents the solution of the boundary value problem of Eqs. (18) and (19). Note that in this problem we needed both sine and cosine terms in the solution. This is because the boundary data were given on $0 \le \theta < 2\pi$ and have period 2π. As a consequence, the full Fourier series is required, rather than sine or cosine terms alone.

10.8

PROBLEMS

1. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, that satisfies the boundary conditions

$$u(0,y) = 0, \qquad u(a,y) = 0, \qquad 0 < y < b,$$
$$u(x,0) = 0, \qquad u(x,b) = g(x), \qquad 0 \le x \le a.$$

(b) Find the solution if

$$g(x) = \begin{cases} x, & 0 \le x \le a/2, \\ a - x, & a/2 \le x \le a. \end{cases}$$

(c) For $a = 3$ and $b = 1$, plot u versus x for several values of y and also plot u versus y for several values of x.

(d) Plot u versus both x and y in three dimensions. Also draw a contour plot showing several level curves of $u(x, y)$ in the xy-plane.

2. Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, that satisfies the boundary conditions

$$u(0,y) = 0, \qquad u(a,y) = 0, \qquad 0 < y < b,$$
$$u(x,0) = h(x), \qquad u(x,b) = 0, \qquad 0 \le x \le a.$$

3. (a) Find the solution $u(x,y)$ of Laplace's equation in the rectangle $0 < x < a$, ... that satisfies the boundary conditions

$$u(0,y) = 0, \qquad u(a,y) = f(y), \qquad 0 < y < b,$$
$$u(x,0) = h(x), \qquad u(x,b) = 0, \qquad 0 \le x \le a.$$

Hint: Consider the possibility of adding the solutions of two problems, one with homogeneous boundary conditions except for $u(a,y) = f(y)$, and the other with homogeneous boundary conditions except for $u(x,0) = h(x)$.

(b) Find the solution if $h(x) = (x/a)^2$ and $f(y) = 1 - (y/b)$.

(c) Let $a = 2$ and $b = 2$. Plot the solution in several ways: u versus x, u versus ..., both x and y, and a contour plot.

4. Show how to find the solution $u(x,y)$ of Laplace's equation in the rectangle ... $0 < y < b$, that satisfies the boundary conditions

$$u(0,y) = k(y), \qquad u(a,y) = f(y), \qquad 0 < y < b,$$
$$u(x,0) = h(x), \qquad u(x,b) = g(x), \qquad 0 \le x \le a.$$

Hint: See Problem 3.

5. Find the solution $u(r,\theta)$ of Laplace's equation

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} = 0$$

outside the circle $r = a$, that satisfies the boundary condition

$$u(a,\theta) = f(\theta), \qquad 0 \le \theta < 2\pi,$$

on the circle. Assume that $u(r,\theta)$ is single-valued and bounded for $r > a$.

6. (a) Find the solution $u(r,\theta)$ of Laplace's equation in the semicircular region ... $0 < \theta < \pi$, that satisfies the boundary conditions

$$u(r,0) = 0, \qquad u(r,\pi) = 0, \qquad 0 \le r < a,$$
$$u(a,\theta) = f(\theta), \qquad 0 \le \theta \le \pi.$$

Assume that u is single-valued and bounded in the given region.

(b) Find the solution if $f(\theta) = \theta(\pi - \theta)$.

(c) Let $a = 2$ and plot the solution in several ways: u versus r, u versus θ, u versus ... and θ, and a contour plot.

7. Find the solution $u(r,\theta)$ of Laplace's equation in the circular sector $0 < r < a$, $0 < \theta < \alpha$, that satisfies the boundary conditions

$$u(r,0) = 0, \qquad u(r,\alpha) = 0, \qquad 0 \le r < a,$$
$$u(a,\theta) = f(\theta), \qquad 0 \le \theta \le \alpha.$$

Assume that u is single-valued and bounded in the sector and that $0 < \alpha < 2\pi$.

8. (a) Find the solution $u(x,y)$ of Laplace's equation in the semi-infinite strip $0 < x < a$, $y > 0$, that satisfies the boundary conditions

$$u(0,y) = 0, \qquad u(a,y) = 0, \qquad y > 0,$$
$$u(x,0) = f(x), \qquad 0 \le x \le a$$

and the additional condition that $u(x,y) \to 0$ as $y \to \infty$.

(b) Find the solution if $f(x) = x(a - x)$.

(c) Let $a = 5$. Find the smallest value of y_0 for which $u(x,y) \le 0.1$ for all $y \ge y_0$.