Math 257/316 Assignment 4 Solutions

1. Consider the heat conduction problem:

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 3,\ t > 0,$$

with homogeneous boundary conditions

$$u(0,t) = u(3,t) = 0.$$

Find the solution for each of the initial conditions (using formulas from class/notes/text if you like):

a) $u(x,0) = 4\sin \pi x$

b) $u(x,0) = \sin(\pi x/3) - 2\sin(2\pi x/3) + 11\sin(2\pi x)$

As worked out in class, the general solution of the PDE and the BCs (taking $\alpha^2 = 5$, $L = 3$) is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/3)e^{-(5n^2\pi^2/9)t},$$

so

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/3).$$

(a)

$$4\sin(\pi x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/3)$$

so we take $c_3 = 4$, and all other coefficients 0:

$$u(x,t) = 4\sin(\pi x)e^{-5\pi^2 t}.$$

(b)

$$\sin(\pi x/3) - 2\sin(2\pi x/3) + 11\sin(2\pi x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/3)$$

so we take $c_1 = 1$, $c_2 = -2$, $c_6 = 11$, and all other coefficients 0:

$$u(x,t) = \sin(\pi x/3)e^{-(5\pi^2/9)t} - 2\sin(2\pi x/3)e^{-(20\pi^2/9)t} + 11\sin(2\pi x)e^{-20\pi^2 t}.$$

2. Use the method of separation of variables to find the most general solution of the following heat conduction problem with "mixed" boundary conditions:

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L,\ t > 0,$$
$$u(0,t) = 0, \quad u_x(L,t) = 0.$$

Separating variables $u(x,t) = X(x)T(t)$ leads to

$$u_t = \alpha^2 u_{xx} \qquad \Longrightarrow \qquad XT' = \alpha^2 X'' T$$

and dividing through by $\alpha^2 XT$,

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} = \text{ constant } =: \lambda$$

(each side must be constant since they depend on different variables). The "X problem"

$$X'' = \lambda X, \quad X(0) = 0, \qquad X'(L) = 0$$

inherits its boundary conditions from those of the original PDE problem (note it is X' not X which should vanish at $x = L$). Cases:

(a) *$\lambda > 0$: the general solution of the ODE is $X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$. $0 = X(0) = c_1 + c_2$ implies $c_2 = -c_1$ so $X(x) = c_1(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}) = 2c_1 \sinh(\sqrt{\lambda}x)$. Then $0 = X'(L) = 2c_1\sqrt{\lambda}\cosh(\sqrt{\lambda}L)$, which can only be satisfied if $c_1 = 0$ (cosh is always positive), so $c_2 = 0$. No non-zero solutions in this case.*

(b) *$\lambda = 0$: then $X = c_1 x + c_2$. $0 = X(0) = c_2$ and $0 = X'(L) = c_1$ shows there are no non-zero solutions in this case either.*

(c) *$\lambda < 0$: the general solution is $X(x) = c_1 \sin(\sqrt{-\lambda}x) + c_2 \cos(\sqrt{-\lambda}x)$. The BCs give $0 = X(0) = c_2$, and $0 = X'(L) = c_1\sqrt{-\lambda}\cos(\sqrt{-\lambda}L)$. So for a non-zero solution, we require*

$$\cos(\sqrt{-\lambda}L) = 0 \qquad \Longrightarrow \qquad \sqrt{-\lambda}L = \pi/2 + k\pi, \quad k = 0, 1, 2, 3, \ldots.$$

So we have our solutions of the X problem:

$$X_k(x) = \sin\left(\frac{\pi/2 + k\pi}{L}x\right), \quad \lambda_k = -\frac{(\pi/2 + k\pi)^2}{L^2}, \quad k = 0, 1, 2, 3, \ldots.$$

The solutions of the corresponding "T problem"

$$T' = \alpha^2 \lambda_k T = -\frac{(\pi/2 + k\pi)^2 \alpha^2}{L^2} T$$

are constant multiples of $e^{-(\pi/2+k\pi)^2\alpha^2 t/L^2}$. *Combining with the corresponding* X, *we have product solutions (of both the heat equation and the given boundary conditions) of the form*

$$e^{-\frac{(\pi/2+k\pi)^2\alpha^2}{L^2}t}\sin\left(\frac{\pi/2+k\pi}{L}x\right), \quad k=0,1,2,3,\ldots.$$

Finally, the most general solution we can write is an infinite linear combination of these (which still satisfies the boundary conditions – since they are homogeneous – as well as the PDE, since it is linear and homogeneous):

$$u(x,t)=\sum_{k=0}^{\infty}c_k e^{-\frac{(\pi/2+k\pi)^2\alpha^2}{L^2}t}\sin\left(\frac{\pi/2+k\pi}{L}x\right).$$

3. Use the method of separation of variables to solve the problem

$$\begin{aligned}&u_t=u_{xx}+au, \quad 0<x<1,\ t>0,\\ &u(0,t)=0,\ \ u(1,t)=0,\\ &u(x,0)=\sin(\pi x)\end{aligned}$$

How does the long term ($t\to\infty$) behaviour of the solution depend on the constant a?

Separating variables $u(x,t)=X(x)T(t)$ *leads to*

$$0=u_t-u_{xx}-au=XT'-X''T-aXT$$

and dividing through by XT *yields*

$$0=\frac{T'}{T}-\frac{X''}{X}-a, \qquad \textit{or} \qquad \frac{T'}{T}-a=\frac{X''}{X}=\textit{const.}=-\lambda,$$

since each side depends on a different variable (note: you could also put the a *on the* X *side – it would all work out the same). The* X *problem,*

$$X''(x)=-\lambda X(x), \quad X(0)=0=X(1),$$

is the same one we encountered in class in solving the heat equation with zero BCs, and we know its solutions:

$$\lambda_n=\pi^2n^2, \quad X_n(x)=\sin(n\pi x), \quad n=1,2,3,\ldots.$$

Then the T *problem, with* $\lambda=\lambda_n$ *is*

$$T'(t)=(a-\lambda_n)T=(a-\pi^2n^2)T,$$

whose solution is (any multiple of)

$$T_n(t) = e^{(a-\pi^2 n^2)t}.$$

The most general solution of the PDE and BCs is an infinite linear combination of the product solutions $X_n(x)T_n(t)$:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{(a-\pi^2 n^2)t}.$$

To satisfy the initial condition we need

$$\sin(\pi x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

and a simple inspection leads to $c_1 = 1$, $c_2 = c_3 = c_4 = \cdots = 0$, so

$$u(x,t) = \sin(\pi x) e^{(a-\pi^2)t}.$$

Looking at the exponential, we see that (for any $0 < x < 1$)

$$\left\{ \begin{array}{cc} \lim_{t\to\infty} u(x,t) = 0 & a < \pi^2 \\ \lim_{t\to\infty} u(x,t) = \infty & a > \pi^2 \\ u(x,t) = \sin(\pi x) \textit{ for all } t & a = \pi^2 \end{array} \right. .$$