Summary of Separation of Variables for heat equation, and practice problems

Case A: zero BC, zero source

$$u(x,t) = \sum_{n=1}^{+\infty} a_n e^{-\lambda_n \alpha^2 t} X_n(x)$$

where λ_n, X_n are eigenvalues and eigenfcns

BC1

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(x,0) = f(x) \\ u(0,t) = 0, \quad u(L,t) = 0 \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n = \sin\left(\frac{n\pi}{L}x\right)$$

$$u(x,t) = \sum_{n=1}^{+\infty} b_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

BC2

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(x,0) = f(x) \\ u_x(0,t) = 0, \quad u_x(L,t) = 0 \end{cases}$$

$$\lambda_0 = 0, \quad X_0 = 1, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n = \cos\left(\frac{n\pi}{L}x\right)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx, \quad n = 0, 1, 2, \dots$$

BC3

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(x,0) = f(x) \\ u_x(0,t) = 0, \; u(L,t) = 0 \end{cases}$$

$$\lambda_n = \left[\frac{2n-1}{2L}\pi\right]^2, \qquad X_n = \cos\left(\frac{2n-1}{2L}\pi x\right)$$

$$u(x,t) = \sum_{n=1}^{+\infty} b_n e^{-\alpha^2\left(\frac{2n-1}{2L}\pi\right)^2 t} \cos\left(\frac{2n-1}{2L}\pi x\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n-1}{2L}\pi x\right), \quad n = 0, 1, 2,$$

BC4

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(x,0) = f(x) \\ u(0,t) = 0, \; u_x(L,t) = 0 \end{cases}$$

$$\lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2, \qquad X_n = \sin\left(\frac{2n-1}{2L}\pi x\right)$$

$$u(x,t) = \sum_{n=1}^{+\infty} b_n e^{-\alpha^2\left(\frac{2n-1}{2L}\pi\right)^2 t} \sin\left(\frac{2n-1}{2L}\pi x\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n-1}{2L}\pi x\right), \quad n = 0, 1, 2,$$

BC5

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(x,0) = f(x) \\ u(-L,t) = u(L,t), \; u_x(-L,t) = u_x(L,t) \end{cases}$$

$$\lambda_0 = 0, \; X_0 = 1, \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \; X_n = c_1\cos\left(\frac{n\pi}{L}x\right) + c_2\sin\left(\frac{n\pi}{L}x\right)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad n = 0, 1, \dots, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Case B : nonzero BC, zero soure

B.1 BC is not Neumann.

$u(x,t) := U(x) + v(x,t)$

where U solves steady-state-problem $\begin{cases} 0 = \alpha^2 U_{xx} \\ U(0) = A, \ U(L) = B \end{cases}$

$v(x,t)$ solves heat equation with 0 BC and 0 source

use Case A to solve $v(x,t)$

B.2. BC is Neumam

$u(x,t) = U(x,t) + v(x,t)$

$U(x,t) = ax^2 + bc + ct$, solves $\begin{cases} U_t = \alpha^2 U_{xx} \\ U_x(0,t) = A, \ U_x(L,t) = B \end{cases}$

use Case A to solve $v(x,t)$

Case C : zero BC, nonzero source

· eigenfunction expansion :

$u(x,t) = \sum u_n(t) X_n(x)$

$s(x,t) = \sum s_n(t) X_n(x)$

$\Rightarrow \quad u_t = \sum u_n'(t) X_n(x), \quad u_{xx} = \sum (-\lambda_n u_n(t)) X_n(x)$

$u_t = -\alpha^2 \lambda_n u_n(t) + S_n(t)$

$\begin{cases} u_n'(t) = \\ u_n(0) = b_n \end{cases}$

Case D: nonzero BC, nonzero source

Step 1: Use B1 or B2 to get rid of inhomogeneous BC.

step 2: Use Case C to solve $v(x,t)$

Case E: time-dependent BC, nonzero source

Step 1: Use B1 or B2 to get rid of inhomoge. BC

Step 2: Use Case C to solve $v(x,t)$

Practice Problems

Ex. 1 Solve

$$\begin{cases} u_t = u_{xx}, & 0 < x < \frac{1}{2} \\ u(x,0) = \sin(2\pi x) \\ u(0,t) = 0, \quad u(\frac{1}{2}, t) = 0 \end{cases}$$

Ex. 2. Solve

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1 \\ u(x,0) = 0 \\ u(0,t) = 1, \quad u(1,t) = 0 \end{cases}$$

Ex. 3 Solve

$$\begin{cases} u_t = 3u_{xx}, & 0<x<1 \\ u(x,0) = 1 \\ u_x(0,t) = 1, \quad u_x(1,t) = 2 \end{cases}$$

Ex. 4 Solve

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin(2\pi x), & 0<x<1 \\ u(x,0) = 0 \\ u(0,t) = 0, \quad u_x(1,t) = 0 \end{cases}$$

Ex. 5 Solve

$$\begin{cases} u_t = u_{xx} - u, & 0<x<1 \\ u(x,0) = 0 \\ u_x(0,t) = 1, \quad u(1,t) = 0 \end{cases}$$

Ex. 6 Solve

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin(\pi x), & 0<x<1 \\ u(x,0) = \sin(2\pi x) \\ u(0,t) = 0, \quad u(1,t) = 0 \end{cases}$$

Ex. 7 Solve

$$\begin{cases} u_t = u_{xx} + x e^{-t}, & 0 < x < 1 \\ u(x,0) = 0 \\ u(0,t) = t, \quad u(1,t) = 2 \end{cases}$$

Ex. 8 Solve

$$\begin{cases} u_t = 5 u_{xx} + 1 \\ u(x,0) = 0 \\ u_x(0,t) = 1, \quad u_x(1,t) = 0 \end{cases}$$

Ex. 9 Solve

$$\begin{cases} u_t = u_{xx} + e^{-t} \\ u(x,0) = 0 \\ u(0,t) = 1, \quad u(1,t) = 2 \end{cases}$$

Ex. 10

$$\begin{cases} u_t = u_{xx} - u + x \\ u(x,0) = 0 \\ u_x(0,t) = 1, \quad u_x(1,t) = 0 \end{cases}$$