Practice Problems for Wave Equation (Lecture 21 - 23, Lecture 8)

Formula:

1) D'Alembert's Formula for

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

2) Separation of Variables for Wave Equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad 0 < x < l \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$$

with

2.1 Dirichlet BC: $u(0,t) = u(l,t) = 0$

$$u(x,t) = \sum_{n=1}^{+\infty} \left(a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t\right) \sin\left(\frac{n\pi}{L} x\right)$$

where

$$f(x) = \sum_{n=1}^{+\infty} a_n \sin\left(\frac{n\pi}{L} x\right)$$

$$g(x) = \sum_{n=1}^{+\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi}{L} x\right)$$

So

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$\frac{n\pi c}{L} b_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

2.2 Neumann BC: $u_x(0,t) = u_x(L,t) = 0$

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t \right) \cos\left(\frac{n\pi}{L} x\right)$$

where

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\left(\frac{n\pi}{L} x\right)$$

$$g(x) = \frac{b_0}{2} + \sum_{n=1}^{+\infty} \frac{n\pi c}{L} b_n \cos\left(\frac{n\pi}{L} x\right)$$

and so

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

$$b_0 = \frac{2}{L} \int_0^L g(x)\, dx$$

$$\frac{n\pi c}{L} b_n = \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

2.3 periodic BC: $u(-L,t) = u(L,t)$, $u_x(-L,t) = u_x(L,t)$

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t \right) \left(c_n \cos \frac{n\pi}{L} x + d_n \sin \frac{n\pi}{L} x \right)$$

where

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n c_n \cos \frac{n\pi}{L} x + a_n d_n \sin \frac{n\pi}{L} x \right)$$

$$g(x) = \frac{b_0}{2} + \sum_{n=1}^{+\infty} \left(\frac{n\pi c}{L} b_n c_n \cos \frac{n\pi}{L} x + \frac{n\pi c}{L} b_n d_n \sin \frac{n\pi}{L} x \right)$$

and so

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n c_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$a_n d_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

$$b_0 = \frac{1}{L}\int_{-L}^{L} g(x)\,dx$$

$$\frac{n\pi c}{L} b_n c_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos\frac{n\pi}{L}x\,dx$$

$$\frac{n\pi c}{L} b_n d_n = \frac{1}{L}\int_{-L}^{L} g(x)\sin\frac{n\pi}{L}x\,dx$$

2.4. MIX 1 and MIX 2 are similar

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3) Finite-difference-method.

$$u_n^{k+1} = 2u_n^k - u_n^{k-1} + \frac{c^2\Delta t^2}{\Delta x^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)$$

$$u_n^0 = f(x_n)$$

$$u_n^1 = u_n^0 + \frac{1}{2}\frac{c^2\Delta t^2}{\Delta x^2}\left(u_{n+1}^0 - 2u_n^0 + u_{n-1}^0\right) + \Delta t\, g(x_n)$$

$t_0 = 0 \rightarrow u_n^0$

$t_1 = \Delta t \rightarrow u_n^1$

4) Relation between 1) and 2) in case of BCs (2.1), (2.2) or (2.3).

$$u(x,t) = \frac{1}{2}\left[f_{ext}(x+ct) + f_{ext}(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{ext}(s)\, ds$$

where f_{ext} is the odd, or even, or periodic extension of $f(x)$. Similarly for g_{ext}.

Practice Problems

1. Solve $u_{tt} = 4u_{xx}$, $u(x,0) = \cos x$, $u_t(x,0) = e^x$

2. Solve $u_{tt} = u_{xx}$, $u(x,0) = \begin{cases} |x|-1, & |x| \le 1 \\ 0, & |x|>1 \end{cases}$, $u_t(x,0) = 0$

   when $t = \frac{1}{2}$, 1 and 2

3. Solve $u_{tt} = 4u_{xx}$, $u(x,0) = 0$,

   $u_t(x,0) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

   for $t = 1$, 2 and 4

4. Solve the wave equation

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < 1 \\ u(x,0) = |x|, & u_t(x,0) = 1 \\ u(0,t) = 0, & u(1,t) = 0 \end{cases}$$

5. Solve the wave equation

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < 1 \\ u(x,0) = |x|. & u_t(x,0) = 1 \\ u_x(0,t) = 0, & u_x(1,t) = 0 \end{cases}$$

6. Solve the wave equation

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < 1 \\ u(x,0) = \sin(2\pi x) & u_t(x,0) = \sin(4\pi x) \\ u(0,t) = 0, & u(1,t) = 0 \end{cases}$$

7. Solve the wave equation

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < 1 \\ u(x,0) = x, & u_t(x,0) = 0 \\ u(0,t) = 1, & u(1,t) = 2 \end{cases}$$

Hint: Let $u(x,t) = U(x) + v(x,t)$ where U is the steady-state

8. Use finite difference method to solve

$$
\begin{cases}
u_{tt} = 4 u_{xx}, & 0 < x < 1 \\
u(x,0) = |x|, & u_t(x,0) = 1 \\
u(0,t) = 0, & u_x(1,t) = 0
\end{cases}
$$

with $\Delta x = 0.1$, $\Delta t = 0.02$

Compute u_n^3 for $n = 1, 2, \cdots, 10$

9. (a) Use finite difference method to solve

$$
\begin{cases}
u_{tt} = u_{xx}, \quad 0 < x < 1 \\
u(x,0) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1-x, & \frac{1}{2} < x < 1 \end{cases}, \quad u_t(x,0) = 0 \\
u(0,t) = 0, \quad u(1,t) = 0
\end{cases}
$$

with $\Delta x = 0.1$, $\Delta t = 0.02$

Compute u_n^3

(b) Use the extension and d'Alembert's formula to compute $u(x, \frac{1}{2})$