Solutions to Practice Problems on Wave Equation

1. Use d'Alembert's Formula:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)\,dx$$

$$= \frac{1}{2}\left[\cos(x+2t) + \cos(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} e^{x}\,dx$$

$$= \cos x \cos 2t + \frac{1}{4}\left(e^{x+2t} - e^{x-2t}\right)$$

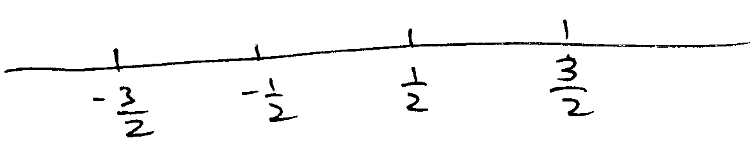
2. Use d'Alembert's formula

$$f(x) = \begin{cases} |x|-1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}, \quad g = 0$$

<u>$t=\frac{1}{2}$</u>: $f(x+\frac{1}{2}) = \begin{cases} |x+\frac{1}{2}|-1, & |x+\frac{1}{2}| \le 1 \\ 0, & |x+\frac{1}{2}| > 1 \end{cases}$

$$= \begin{cases} |x+\frac{1}{2}|-1, & -\frac{3}{2} \le x \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$f(x-\tfrac{1}{2}) = \begin{cases} |x-\frac{1}{2}|-1, & -\frac{1}{2} \le x \le \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$



For $x < -\frac{3}{2}$ or $x > \frac{3}{2}$

$$u(x, \tfrac{1}{2}) = 0$$

For $-\frac{3}{2} < x < -\frac{1}{2}$, $\quad f(x+\frac{1}{2}) = -(x+\frac{1}{2}) - 1$

$$f(x-\tfrac{1}{2}) = -(x-\tfrac{1}{2}) - 1$$

So $u(x, \frac{1}{2}) = \frac{1}{2}\left[-(x+\frac{1}{2}) - 1 + (x-\frac{1}{2}) - 1\right]$

$$= -x - 1$$

For $-\frac{1}{2} < x < \frac{1}{2}$, $\quad f(x+\frac{1}{2}) = x + \frac{1}{2} - 1$

$$f(x-\tfrac{1}{2}) = -(x-\tfrac{1}{2}) - 1$$

$$u(x, \tfrac{1}{2}) = \tfrac{1}{2}\left[x + \tfrac{1}{2} - 1 - (x-\tfrac{1}{2}) - 1\right]$$

$$= -\tfrac{1}{2}$$

For $\frac{1}{2} < x < \frac{3}{2}$, $\quad f(x+\frac{1}{2}) = x + \frac{1}{2} - 1$

$$f(x-\tfrac{1}{2}) = (x-\tfrac{1}{2}) - 1$$

$$u(x, \tfrac{1}{2}) = x - 1$$

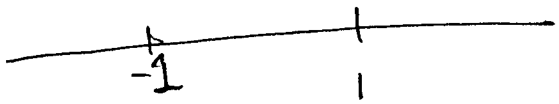
The computations at other places are similar

3. Here $c = 2$. $\quad f = 0, \quad g = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

So $u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} g = \frac{1}{4}$ length of $[x-2t, x+2t] \cap [-1, 1]$

Let us consider $\underline{t=2}$.

$u(x,2) = \frac{1}{4}$ length of $[x-4,\ x+4] \cap [-1, 1]$



If $x+4 < -1$, then $[x-4, x+4] \cap [-1,1] = \phi$, $\Rightarrow u = 0$

If $1 > x+4 > -1$. ~~x-4<-1~~ then $x-4<1$, $[x-4, x+4] \cap [-1, 1]$

$= [-1, x+4]$

$\Rightarrow u = \frac{1}{4}(x+5)$

If $x+4>1$, ~~x-4>~~ $-1 < x-4 < 1$ $\Rightarrow$ $[x-4, x+4] \cap [-1, 1]$

$= [x-4, 1]$

$u = \frac{1}{4}(5-x)$

If $x-4 > 1 \Rightarrow u = 0$

4. $u(x,t) = \sum_{n=1}^{+\infty} \left(a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t\right) \sin\left(\frac{n\pi}{L} x\right)$

$L=1$, $c=1$, so

$u(x,t) = \sum (a_n \cos n\pi t + b_n \sin n\pi t) \sin(n\pi x)$

$u(x,0) = |x| = \sum a_n \sin n\pi x \Rightarrow a_n = 2\int_0^1 |x| \sin n\pi x \, dx$

$u_t(x,0) = 1 = \sum n\pi b_n \sin n\pi x \Rightarrow n\pi b_n = 2\int_0^1 \sin n\pi x \, dx$

5. $u(x,t) = \frac{a_0 + b_0 t}{2} + \sum \left(a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi}{L} t\right) \cos \frac{n\pi}{L} x$

$L=1, \quad c=2, \quad f=|x|, \quad g=1$

So

$u(x,0) = |x| = \frac{a_0}{2} + \sum a_n \cos n\pi x \quad \Rightarrow$

$a_n = 2 \int_0^1 |x| \cos n\pi x, \quad n=0,1,2,\ldots$

$u_t(x,0) = \frac{b_0}{2} + \sum (b_n n\pi \cos n\pi x) = 1$

$b_0 = 2 \int_0^1 1\, dx$

$n\pi b_n = 2 \int_0^1 1 \cos n\pi x\, dx$

6. $c=2$, Dirichlet, $L=1$

$u(x,t) = \sum (a_n \cos 2n\pi t + b_n \sin 2n\pi t) \sin(n\pi x)$

$u(x,0) = \sin(2\pi x) = \sum a_n \sin(n\pi x) \Rightarrow a_n = \begin{cases} 1, & n=2 \\ 0, & n \neq 2 \end{cases}$

$u_t(x,0) = \sin 4\pi x = \sum b_n \cdot 2n\pi \sin(n\pi x) \Rightarrow b_n = \begin{cases} \frac{1}{2n\pi}, & n=4 \\ 0, & n \neq 4 \end{cases}$

Hence

$u(x,t) = \cos 4\pi t \sin 2\pi x + \frac{1}{8\pi} \sin 8\pi t \sin(4\pi x)$

7. Let U be the steady-state $\begin{cases} 0 = 4U_{xx} \\ U(0)=1, \ U(1)=2 \end{cases}$

$\Rightarrow U(x) = 1 + x$

Let $u = U(x) + v(x,t)$. Then $v(x,t)$ satisfies

$$\begin{cases} v_{tt} = 4 v_{xx} \\ v(x,0) = -1, \quad v_t(x,0) = 0 \\ v(0,t) = 0, \quad v(1,t) = 0 \end{cases}$$

$$v(x,t) = \sum a_n \cos 2n\pi t \sin n\pi x$$

$$a_n = 2 \int_0^1 (-1) \sin(n\pi x)\, dx$$

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8. $f(x) = |x|$, $c = 2$, $g = 1$, $\frac{4\Delta t^2}{\Delta x^2} = \frac{4 \times 0.02^2}{0.1^2} = \frac{0.0016}{0.01} = 0.16$

Iteration:

$$u_n^{k+1} = 2u_n^k - u_n^{k-1} + \frac{4\Delta t^2}{\Delta x^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)$$

$$u_n^{k+1} = 2u_n^k - u_n^{k-1} + 0.16\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)$$

IC: $u_n^0 = f(n\Delta x) = n\Delta x$

$u_n^1 - u_n^{-1} = 2\Delta t\, g(n\Delta x) = 0.04$

BCs: $u_0^k = 0$

Let us compute u_n^1:

$$\begin{cases} u_n^1 - u_n^{-1} = 0.04 \\ u_n^1 = 2u_n^0 - u_n^{-1} + 0.16\left(u_{n+1}^0 - 2u_n^0 + u_{n-1}^0\right) \end{cases}$$

$$\Rightarrow u_n^1 = u_n^0 + 0.08\left(u_{n+1}^0 - 2u_n^0 + u_{n-1}^0\right) + 0.02$$
$$= 0.08\left(u_{n+1}^0 + u_{n-1}^0\right) + 0.84\, u_n^0 + 0.02$$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11} (ghost)	
$t_0=0$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	0.9	u_n^0
$t_0=t_1$	0	0.12	0.22	0.32	0.42	0.52	0.62	0.72	0.82	0.92	1.004	0.92	u_n^1
	0	0.4536	0.24	0.34	0.44	0.54	0.64	0.74	0.84	0.937	0.98112	0.937	u_n^2

For u_n^1, we use $\quad u_n^1 = 0.08\left(u_{n+1}^0 + u_{n-1}^0\right) + 0.84\, u_n^0 + 0.02$

For u_n^2, we use

$$u_n^2 = 2u_n^1 - u_n^0 + 0.16\left(u_{n+1}^1 - 2u_n^1 + u_{n-1}^1\right)$$
$$= 0.16\left(u_{n+1}^1 + u_{n-1}^1\right) + 1.68\, u_n^1 - u_n^0$$

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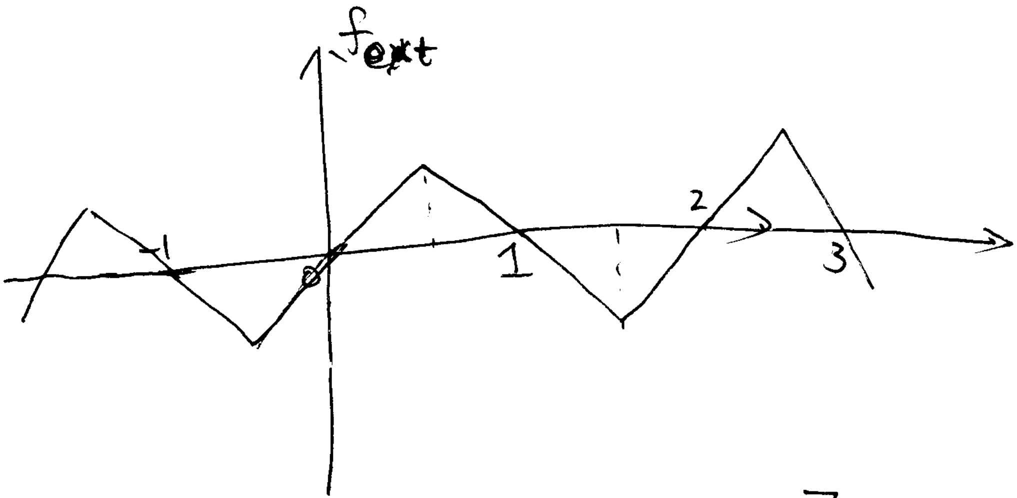
9 (a) the computation is similar to Problem 8

(b). Since we have Dirichlet BC, we extend $f(x)$, $g(x)$, oddly and then periodically.

So



$$u(x, \tfrac{1}{2}) = \tfrac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

$$= \tfrac{1}{2}\left[f_{ext}(x+\tfrac{1}{2}) + f_{ext}(x-\tfrac{1}{2})\right]$$

For $0 < x < 1$, $\quad \frac{1}{2} < x + \frac{1}{2} < \frac{3}{2}$

$$f(x+\tfrac{1}{2}) = 1 - (x+\tfrac{1}{2}) = \tfrac{1}{2} - x$$

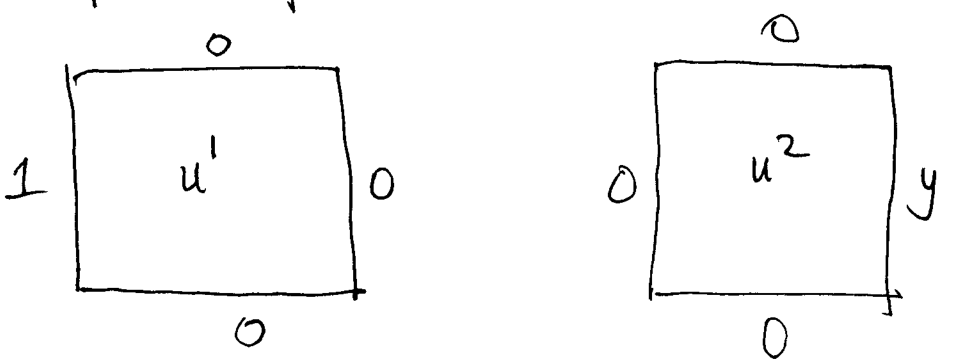
$$-\tfrac{1}{2} < x - \tfrac{1}{2} < \tfrac{1}{2}$$

$$f(x-\tfrac{1}{2}) = x - \tfrac{1}{2}.$$

So $\quad u(x, \frac{1}{2}) = \frac{1}{2}\left[\frac{1}{2} - x + x - \frac{1}{2}\right] = 0.$

Solutions to Practice problems for Laplace Equation:

1. decompose the problem into two:



For u': $Y'' + \lambda Y = 0, \quad Y(0) = Y(\pi) = 0 \quad \Rightarrow \lambda = \left(\frac{n\pi}{\pi}\right)^2, \; Y = \sin ny$

$X'' - \lambda X = 0, \quad X(\pi) = 0 \quad \Rightarrow X = \sinh(n(x-\pi))$

$$u'(x,y) = \sum_{n=1}^{+\infty} A_n \sinh(n(x-\pi)) \sin ny$$

$$u'(0,y) = 1 \Rightarrow \quad 1 = \sum_{n=1}^{+\infty} A_n \sinh(-n\pi) \sin ny$$

$$-A_n \sinh(n\pi) = \frac{2}{\pi} \int_0^{\pi} \sin ny \, dy$$

For u^2: $X'' - \lambda X = 0, \quad X(0) = 0 \Rightarrow \quad X = \sinh(nx)$

$$u^2(x,y) = \sum B_n \sinh(nx) \sin ny$$

$$u^2(\pi,y) = y = \sum B_n \sinh(n\pi) \sin ny$$

$$B_n \sinh(n\pi) = \frac{2}{\pi} \int_0^{\pi} y \sin ny \, dy$$

2. $Y''+\lambda Y=0,\ Y'(0)=Y'(\pi)=0 \Rightarrow \lambda_n=n^2,\ Y_n=\cos ny$

$X''-\lambda X=0,\ X(0)=0 \Rightarrow X=\sinh(nx)$

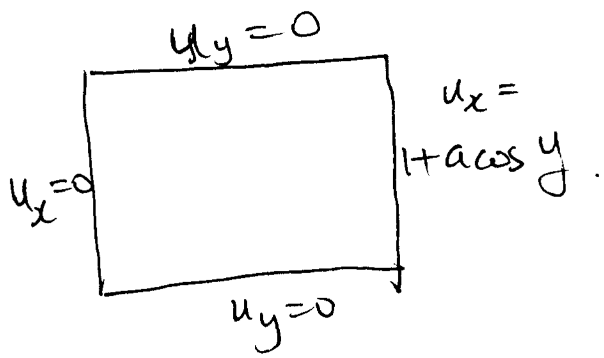
For $n=0$, $X=x$.

So
$$u=\frac{A_0x}{2}+\sum_{n=1}^{+\infty}A_n\sinh(nx)\cos ny$$
$$u(\pi,y)=\frac{1}{2}(1+\cos 2y)=\frac{A_0\pi}{2}+\sum_{n=1}^{+\infty}A_n\sinh(n\pi)\cos ny$$

So $A_0\pi=1$, $A_2\sinh(2\pi)=\frac{1}{2}$, $A_n\not\equiv 0$ for $n\neq 0,2$

$$u(x,y)=\frac{x}{2\pi}+\frac{1}{2}\sinh(2x)\cos 2y$$

3.



$Y''+\lambda Y=0,\ Y'(0)=Y'(\pi)=0 \Rightarrow \lambda_n=n^2,\ Y_n=\cos ny$

$X''-\lambda X=0,\ X'(0)=0 \Rightarrow X=\cosh(nx)$.

So
$$u=\frac{A_0}{2}+\sum_{n=1}^{+\infty}A_n\cosh(nx)\cos ny$$
$$u_x=1+a\cos y=\sum_{n=1}^{+\infty}A_n n\sinh(n\pi)\cos ny$$

A necessary condition is

$$\int_0^{\pi} (1+a\cos y)\,dy = 0$$

which never holds

So there is no solution, for all a.

4. $X'' + \lambda X = 0, \quad X(0)=0,\ X'(a)=0$

$$Y'' - \lambda Y = 0$$

MIX1 $\Rightarrow \lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2 = \left(\frac{2n-1}{2a}\pi\right)^2$

$$X_n(x) = \sin\left(\frac{2n-1}{2a}\pi x\right)$$

$$Y'' - \lambda_n Y = 0 \Rightarrow Y = c_1 e^{-\sqrt{\lambda_n}\,y} + c_2 e^{\sqrt{\lambda_n}\,y}$$

$$Y \text{ is bdd} \Rightarrow Y = c_1 e^{-\sqrt{\lambda_n}\,y}$$

so $$u = \sum_{n=1}^{+\infty} A_n e^{-\sqrt{\lambda_n}\,y} \sin(\sqrt{\lambda_n}\,x)$$

$$u(x,0) = x = \sum A_n \sin\left(\frac{2n-1}{2a}\pi x\right)$$

$$A_n = \frac{2}{a}\int_0^a x \sin\left(\frac{2n-1}{2a}\pi x\right)dx$$

5. $$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} f(\theta)\sin n\theta \, d\theta$$

Now $f(\theta) = 1 + 3\sin 2\theta$ so

$$u(2,\theta) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} 2^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$= 1 + 3\sin 2\theta$$

$$\Rightarrow a_0 = 2, \quad 2^2 b_2 = 3, \quad \text{all others} = 0$$

$$u(r,\theta) = 1 + \frac{3}{4} r^2 \sin 2\theta$$

6. $$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = 0, \quad \Theta'(\pi) = 0$$

$$\Rightarrow \lambda = \left(\frac{2n-1}{2L}\pi\right)^2 = \left(\frac{2n-1}{2\pi}\pi\right)^2 = \left(n - \frac{1}{2}\right)^2$$

$$\Theta = \sin\left(n - \frac{1}{2}\right)\theta$$

$$R'' + \frac{1}{r}R' - \frac{\lambda^2}{r^2}R = 0 \Rightarrow R = c_1 r^{\sqrt{\lambda}} + c_2 r^{-\sqrt{\lambda}}$$

$$R \text{ is bdd} \Rightarrow R = c_1 r^{\sqrt{\lambda}} = c_1 r^{n-\frac{1}{2}}$$

So $$u = \sum_{n=1}^{+\infty} a_n r^{n-\frac{1}{2}} \sin\left(n - \frac{1}{2}\right)\theta$$

$$u(1,\theta) = \sin\theta = \sum_{n=1}^{+\infty} a_n \sin(n-\tfrac{1}{2})\theta$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin\theta \sin(n-\tfrac{1}{2})\theta \, d\theta$$

$$= \cdots$$

7. annulus

$$u(r,\theta) = \frac{a_0 + b_0 \log r}{2} + \sum_{n=1}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta)(c_n r^n + d_n r^{-n})$$

$$u(1,\theta) = \sin\theta = \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta)(c_n + d_n)$$

$\Rightarrow \quad a_0 = 0, \quad a_n(c_n + d_n) = 0, \quad b_n(c_n + d_n) = 0$ for $n \neq 1$

$b_1(c_1 + d_1) = 1$

$$u(2,\theta) = \cos\theta = \frac{a_0 + b_0 \log 2}{2} + \sum_{n=1}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta)(c_n 2^n + d_n 2^{-n})$$

$\Rightarrow \quad a_0 + b_0 \log 2 = 0$

$a_n(c_n 2^n + d_n 2^{-n}) = 0$ except $n=1$

$b_n(c_n 2^n + d_n 2^{-n}) = 0, \ \forall n$

$b_1(c_1 + d_1) = 1, \quad b_1(c_1 2 + d_1 2^{-1}) = 0$

Thus $a_0 = b_0 = 0$. $\quad a_1(c_1 2 + d_1 2^{-1}) = 1, \quad a_1(c_1 + d_1) = 0$.

We solve from

$$\left.\begin{aligned} b_1(c_1+d_1) &= 1 = b_1c_1 + b_1d_1 \\ b_1(c_1 2 + d_1 2^{-1}) &= 0 = 2b_1c_1 + 2^{-1} b_1 d_1 \end{aligned}\right\} \Rightarrow \begin{aligned} b_1c_1 &= -\tfrac{1}{3} \\ b_1 d_1 &= \tfrac{4}{3} \end{aligned}$$

$$\left.\begin{aligned} a_1(2c_1 + 2^{-1}d_1) = 1 &\Rightarrow 2a_1c_1 + 2^{-1}a_1 d_1 = 1 \\ a_1c_1 + a_1d_1 = 0 &\Rightarrow a_1c_1 + a_1d_1 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a_1c_1 &= \tfrac{2}{3} \\ a_1d_1 &= -\tfrac{2}{3} \end{aligned}$$

So

$$u = (a_1\cos\theta + b_1\sin\theta)(c_1 r + d_1 r^{-1})$$

$$= \cancel{r\tfrac{4}{3}\cos\theta + r\sin\theta}$$

$$= \left(\tfrac{2}{3}r - \tfrac{2}{3}r^{-1}\right)\cos\theta + \left(-\tfrac{1}{3}r + \tfrac{4}{3}r^{-1}\right)\sin\theta$$

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8. ~~Sol~~ $u(r,\theta) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{+\infty} \cancel{a_n}\, r^{-n}(a_n\cos n\theta + b_n \sin n\theta)$

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n\cos n\theta + b_n\sin n\theta)$$

$$= \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

So $a_0 = 1$, $a_2 = \frac{1}{2}$, all others are 0.

$$u(r,\theta) = \frac{1}{2} + r^{-2}\cdot\frac{1}{2}\cos 2\theta$$

9. $R'' + \frac{1}{r}R - \frac{\lambda}{r^2}R = 0$, $\Theta'' + \lambda\Theta = 0$

$\Theta(0) = 0$, $\Theta(\frac{\pi}{2}) = 0$

so $\lambda = \left(\frac{n\pi}{\frac{\pi}{2}}\right)^2 = 4n^2$, $\Theta = \sin(4n\theta)$

$R = c_1 r^{4n} + c_2 r^{-4n}$

So $u(r,\theta) = \sum_{n=1}^{+\infty} \left(a_n r^{4n} + b_n r^{-4n}\right) \sin(4n\theta)$

$u(1,\theta) = \sum_{n=1}^{+\infty} (a_n + b_n) \sin 4n\theta = 1$

$\Rightarrow a_n + b_n = \frac{2}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 1 \sin 4n\theta \, d\theta = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin 4n\theta \, d\theta$

$= \frac{1}{n\pi}$ — (1)

$u(2,\theta) = \sum_{n=1}^{+\infty} \left(a_n 2^{4n} + b_n 2^{-4n}\right) \sin(4n\theta)$

$= \sin\theta$

$a_n 2^{4n} + b_n 2^{-4n} = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin\theta \sin 4n\theta \, d\theta$ —(2)

$= \frac{1}{\pi} \left(-\frac{1}{4n-1} - \frac{1}{4n+1} \right)$

From (1) and (2) $\Rightarrow$

$a_n = \dfrac{-\frac{1}{\pi}\left(\frac{1}{4n-1} + \frac{1}{4n+1}\right) - \frac{1}{n\pi}}{2^{8n} - 1}$

$b_n = \dfrac{\frac{1}{n\pi} + \frac{1}{\pi}\left(\frac{1}{4n-1} + \frac{1}{4n-1}\right)}{1 - 2^{-8n}}$

Solutions for (S-L-P)

1. $\alpha_1 = -1$, $\alpha_2 = +1$, $\beta_1 = 1$, $\beta_2 = 0$ $\Rightarrow$ $\lambda > 0$.

Let $\lambda = \beta^2$. $\quad y'' + \beta^2 y = 0$

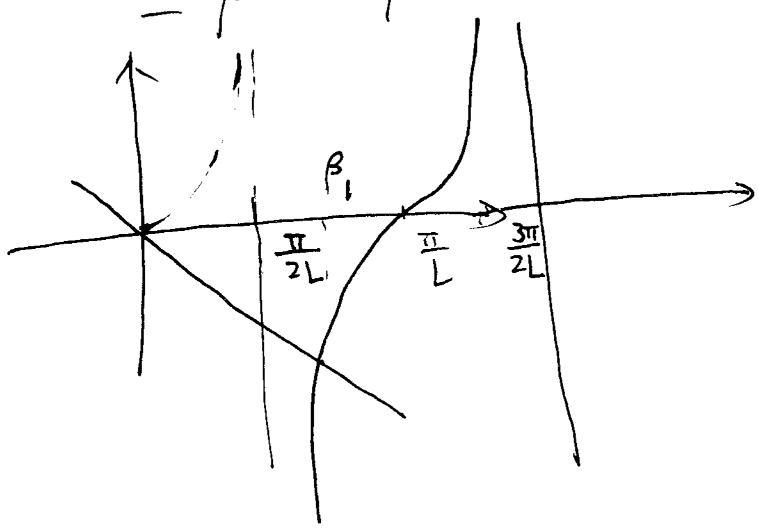
$$y = c_1 \cos\beta x + c_2 \sin\beta x$$

$$\left.\begin{aligned} y'(0) - y(0) = 0 &\Rightarrow c_2\beta - c_1 = 0 \\ y(L) = 0 &\Rightarrow c_1\cos\beta L + c_2 \sin\beta L = 0 \end{aligned}\right\} \; c_1 = c_2\beta$$

$$c_2(\beta\cos\beta L + \sin\beta L) = 0$$

$$\Rightarrow \quad \beta + \tan\beta L = 0$$

$$-\beta = \tan\beta L$$



$$\frac{2n-1}{2L}\pi < \beta_n < \frac{(2n+1)\pi}{2L}, \quad n = 1, 2, \cdots$$

2. $\alpha_1 = -1,\ \alpha_2 = 1,\ \beta_1 = 1,\ \beta_2 = 1 \Rightarrow \lambda = \beta^2 > 0$

$y = c_1 \cos\beta x + c_2 \sin\beta x$

$y'(0) - y(0) \Rightarrow c_2\beta - c_1 = 0 \Rightarrow c_1 = c_2\beta$

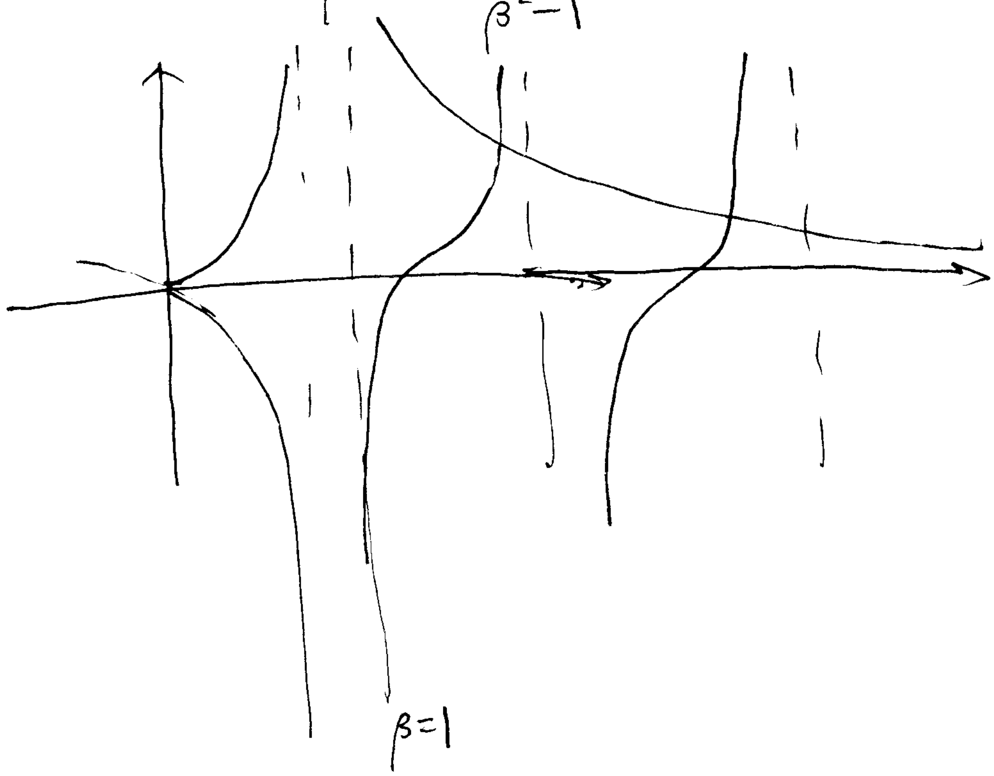
$y'(L) + y(L) = 0 \Rightarrow \beta(-c_1 \sin\beta L + c_2 \cos\beta L) + c_1 \cos\beta L + c_2 \sin\beta L = 0$

So β satisfies

$$\beta(-\beta \sin\beta L + \cos\beta L) + \beta\cos\beta L + \sin\beta L = 0$$

$$(-\beta^2 + 1)\sin\beta L + 2\beta\cos\beta L = 0$$

$$\tan\beta L = \frac{2\beta}{\beta^2 - 1}$$



3. Multiplying by $\mu(x)$

$$\mu(x)\, x\, y'' + 2\mu(x)\, y + \lambda\, \mu(x)\, x\, y = 0$$

Comparing with

$$p y'' + p' y \quad + \lambda r y = 0$$

So $\quad \mu x = p \;,\; 2\mu = p' ,\quad \mu x = r$

so $\frac{p'}{2} \cdot x = p \;\Rightarrow\; \frac{p'}{p} = \frac{2}{x}$

$$\Rightarrow \ln p = 2 \ln x$$

$$\Rightarrow p = x^2$$

$$\mu = \frac{p}{x} = x \;,\quad r = \mu x = x^2$$

So $\quad (x^2 y')' + \lambda x^2 y = 0$

4. $\mu y'' + 2\mu x y + \lambda \mu y = 0$

$$p y'' + p' y' + \lambda r y = 0$$

$$\mu = p ,\quad 2\mu = p' ,\quad \mu = r$$

$$\Rightarrow \; 2p = p' \Rightarrow p = e^{2x} ,\quad \mu = e^{2x} ,\; r = e^{2x}$$

$$(e^{2x} y')' + \lambda e^{2x} y = 0$$

5. Let $\lambda_n = \beta_n^2$ be the eigenvalues in Problem 1.

$$X_n = c_1 \cos\beta_n x + c_2 \sin\beta_n x$$

$$= c_2 (\beta_n \cos\beta_n x + \sin\beta_n x).$$

take $c_2 = 1$

Then $T' + \alpha^2 \lambda_n t = 0 \Rightarrow T = c e^{-\alpha^2 \lambda_n t}$

$$u(x,t) = \sum_{n=1}^{+\infty} A_n e^{-\alpha^2 \lambda_n t} X_n(x)$$

$$u(x,0) = f(x) = \sum_{n=1}^{+\infty} A_n X_n(x)$$

$$A_n = \frac{\int_0^L f(x) X_n(x)\, dx}{\int_0^L X_n^2(x)\, dx}$$

6. ~~u(x,t)~~ $T'' + c^2 \lambda_n T = 0$

$$T = c_1 \cos\beta_n ct + c_2 \sin\beta_n ct$$

$$u(x,t) = \sum_{n=1}^{+\infty} (a_n \cos\beta_n ct + b_n \sin\beta_n ct) X_n(x)$$

$$u(x,0) = f(x) = \sum_{n=1}^{+\infty} a_n X_n(x) \Rightarrow a_n = \frac{\int_0^L f(x) X_n(x)\, dx}{\int_0^L X_n^2}$$

$$u_t(x,0) = g(x) = \sum \beta_n c\, b_n X_n(x)$$

$$\Rightarrow \beta_n c b_n = \frac{\int_0^L g(x) X_n(x)\, dx}{\int_0^L X_n^2}$$

Case 1 $\lambda < \frac{1}{4}$

$$r_1 = \frac{-1-\sqrt{1-4\lambda}}{2}, \quad r_2 = \frac{-1+\sqrt{1-4\lambda}}{2}$$

$$y = c_1 x^{r_1} + c_2 x^{r_2}$$

$$y(1)=0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$y(2)=0 \Rightarrow c_1 2^{r_1} + c_2 2^{r_2} = 0$$

$$c_1 (2^{r_1} - 2^{r_2}) = 0 \Rightarrow c_1 = 0, \text{ impossible}$$

Case 2 $\lambda = \frac{1}{4}$

$$r_1 = -\frac{1}{2} = r_2$$

$$y = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x$$

$$y(1) = 0 \Rightarrow c_1 = 0$$

$$y(2) = 0 \Rightarrow c_2 2^{-\frac{1}{2}} \ln 2 = 0 \Rightarrow c_2 = 0$$

Case 3. $\lambda > \frac{1}{4}$

$$r_1 = -\frac{1}{2} + i\frac{\sqrt{4\lambda-1}}{2}$$

$$y = c_1 x^{-\frac{1}{2}} \cos\left(\frac{\sqrt{4\lambda-1}}{2} \ln x\right) + c_2 x^{-\frac{1}{2}} \sin\left(\frac{\sqrt{4\lambda-1}}{2} \ln x\right)$$

$$y(1)=0 \Rightarrow c_1 = 0$$

$$y(2)=0 \Rightarrow c_2 2^{-\frac{1}{2}} \sin\left(\frac{\sqrt{4\lambda-1}}{2} \ln 2\right) = 0$$

7. $\begin{cases} X'' + \lambda X = 0 \\ X'(0) - X(0) = 0, \quad X(a) = 0 \end{cases}$ $\qquad \begin{cases} Y'' - \lambda Y = 0 \\ Y(0) = 0 \end{cases}$

So $\lambda = \beta_n^2$ is the eigenvalue found in problem 1.

Let $X_n = \beta_n \cos \beta_n x + \sin \beta_n x$ be the eigenfunction

Then $Y_n = \sinh \beta_n y$

$$u = \sum_{n=1}^{+\infty} A_n X_n(x) \sinh \beta_n y$$

$u(x, b) = g(x)$

$$\Rightarrow \quad g(x) = \sum_{n=1}^{+\infty} A_n \sinh \beta_n b \; X_n(x)$$

$$A_n \sinh \beta_n b = \frac{\int_0^L X_n(x) g(x)}{\int_0^L X_n^2}$$

8. This problem has no sol'n. Let us change it to

$\begin{cases} (x^2 y')' + \lambda y = 0, & 1 < y < 2 \\ y(1) = 0, & y(2) = 0 \end{cases}$

$\lambda > 0.$ $\qquad x^2 y'' + 2x y' + \lambda y = 0$

$y = x^r \qquad r(r-1) + 2r + \lambda = 0$

$r^2 + r + \lambda = 0.$

Case 1. $\lambda < \frac{1}{4}$. Then

$$r_1 = \frac{-1 - \sqrt{1-4\lambda}}{2} < 0, \quad r_2 = \frac{-1+\sqrt{1-4\lambda}}{2} < 0$$

$$y = c_1 x^{r_1} + c_2 x^{r_2}$$

$$\left.\begin{array}{ll} y'(1) - y(1) = 0 \Rightarrow & r_1 c_1 + r_2 c_2 - (c_1 + c_2) = 0 \\ y(2) = 0 \Rightarrow & c_1 2^{r_1} + c_2 2^{r_2} = 0 \end{array}\right\}$$

$$(r_1 - 1) c_1 + (r_2 - 1) c_2 = 0$$

$$c_1 2^{r_1} + c_2 2^{r_2} = 0. \Rightarrow c_2 = -c_1 2^{r_1 - r_2}$$

$$(r_1 - 1) - (r_2 - 1) 2^{r_1 - r_2} = 0 \quad \Rightarrow \text{impossible}$$

Case 2 $\lambda = \frac{1}{4}$. $\quad r_1 = r_2 = -\frac{1}{2}$

$$y = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x$$

This is impossible

Case 3 $\lambda > \frac{1}{4}$, $\quad r_1 = -\frac{1}{2} + \frac{\sqrt{4\lambda - 1}}{2} i$

So $y = x^{-\frac{1}{2}} \cos\left(\frac{\sqrt{4\lambda-1}}{2} \ln x\right) + x^{-\frac{1}{2}} \sin\left(\frac{\sqrt{4\lambda-1}}{2} \ln x\right)$

$y'(1) - y(1) = 0 \quad \Rightarrow$

$$\Rightarrow \frac{\sqrt{4\lambda-1}}{2} \ln 2 = n\pi$$

$$4\lambda - 1 = \left(\frac{2n\pi}{\ln 2}\right)^2$$

$$\lambda_n = \frac{1}{4} + \frac{1}{4}\left(\frac{2n\pi}{\ln 2}\right)^2$$

$$y_n(x) = x^{-\frac{1}{2}} \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$$

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9. Step 1: Solve steady-state-problem

$$U'' = 0$$

$$U'(0) - U(0) = A, \quad U(L) = B$$

$$U = C_1 + C_2 x \Rightarrow \quad C_2 - C_1 = A, \quad C_1 + C_2 L = B$$

$$C_2 = \frac{B+A}{L+1}, \quad C_1 = \frac{B - AL}{L+1}$$

Step 2. $u = U(x) + v(x,t)$

$$\begin{cases} v_t = \alpha^2 v_{xx} \\ v(x,0) = f(x) - U(x) \\ v_x(0,t) - v(0,t) = 0, \quad v(L,t) = 0 \end{cases}$$

Solve it as in Problem 5.

10. This problem has no sol'n. Let us change it to

$$\begin{cases} u_{tt} = c^2(x^2 u_x)_x, & 1<x<2 \\ u(x,0)=f(x), & u_t(x,0)=g(x) \\ u(1,t)=0, & u(2,t)=0 \end{cases}$$

step 1. $u = X(x)T(t)$

$$(x^2X')' + \lambda X = 0, \quad T'' + c^2\lambda T = 0$$

step 2 (EVP) $\begin{cases} X(1)=X(2)=0 \\ (x^2X')'+\lambda X = 0 \end{cases}$ $\quad T''+c^2\lambda T = 0$

step 3
(EVP) has been solved in Problem 8

$$\lambda_n = \frac{1}{4} + \frac{1}{4}\left(\frac{2n\pi}{\ln 2}\right)^2$$

$$X_n = x^{-\frac{1}{2}} \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$$

(ODE): $T = c_1 \cos c\sqrt{\lambda_n}\, t + c_2 \sin c\sqrt{\lambda_n}\, t$

step 4. $u(x,t) = \sum \left(a_n \cos(\sqrt{\lambda_n}\, ct) + b_n \sin(\sqrt{\lambda_n}\, ct)\right) X_n(x)$

$u(x,0) = f(x) = \sum a_n X_n(x) \Rightarrow a_n = \dfrac{\int_1^2 f(x) X_n}{\int_1^2 X_n^2}$

$u_t(x,0) = g(x) \Rightarrow \sum \sqrt{\lambda_n}\, c\, b_n X_n = g \Rightarrow \sqrt{\lambda_n}\, c\, b_n = \dfrac{\int_1^2 g(x) X_n}{\int_1^2 X_n^2}$