Sol'ns to Assignment One:

1. Let $f(z) = \dfrac{e^z}{\sin^3 z}$. It has one pole inside $|z|=1$

$$f(z) = \frac{e^z}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^3} = \frac{1 + \frac{z}{\cancel{2}} + \frac{z^2}{2} + \cdots}{z^3\left(1 - \frac{z^2}{6} + \frac{z^4}{30} - \cdots\right)^3}$$

$$= \frac{1}{z^3}\left(1 + z + \frac{z^2}{2} + \cdots\right)\left(1 - \frac{z^2}{6} + \frac{z^4}{30} + \cdots\right)^{-3}$$

$$= \left(\frac{1}{z^3}\right)\left(1 + z + \frac{z^2}{2} + \cdots\right)\left(1 + \frac{z^2}{2} + o(z^4)\right)$$

$$= \frac{1}{z^3}\left(\cdots + z^2 + \cdots\right)$$

So $\operatorname{Res}(f, 0) = 1$

$$\int_{|z|=1} \frac{e^z}{\sin^3 z}\,dz = 2\pi i \cdot 1 = 2\pi i$$

2. (a). S~~ince~~ $z = e^{i\varphi}$, $dz = i e^{i\varphi}\, d\varphi \Rightarrow d\varphi = \dfrac{dz}{iz}$

$$2 + \sin\varphi = 2 + \frac{z - \frac{1}{z}}{2i}. \text{ So}$$

$$\int_0^{2\pi} \frac{d\varphi}{2 + \sin\varphi} = \int_{|c|=1} \frac{1}{2 + \frac{z - \frac{1}{z}}{2i}} \cdot \frac{dz}{iz}$$

$$= \int_{|c|=1} \frac{2\,dz}{z^2 + 4iz - 1}$$

$$f(z) = \frac{2}{z^2+4iz-1}$$

poles : $z^2 + 4iz - 1 = 0$

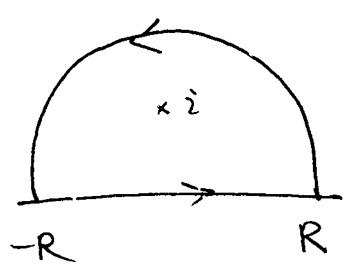
$$z = \frac{-4i \pm \sqrt{(4i)^2+4}}{2} = -2i \pm \sqrt{-\cancel{3}} = -2i \pm \sqrt{3}\, i$$

$$z_1 = (-2+\sqrt{3})\, i \,, \quad z_2 = (-2-\sqrt{3})\, i$$

Thus $\displaystyle \operatorname{Res}(f, z_1) = \frac{2}{2z_1+4i} = \frac{2}{2(-2+\sqrt{3})i+4i} = \frac{1}{\sqrt{3}\, i} = -\frac{i}{\sqrt{3}}$

$$\int_{|z|=1} \frac{2\, dz}{z^2+4iz-1} = 2\pi i \cdot \left(-\frac{i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}} = \int_0^{2\pi} \frac{d\varphi}{2+\sin\varphi}$$

(b). $\displaystyle f(z) = \frac{e^{iz}}{(1+z^2)^2}$



$\deg Q = 4 \geq \deg P + 1 = 0+1$

So $\displaystyle \int_{C_R} f(z) \to 0$ as $R \to +\infty$.

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{(1+x^2)^2}\, dx = 2\pi i \operatorname{Res}(f, i).$$

$$f(z) = \frac{e^{iz}}{(z+i)^2 (z-i)^2} \,, \quad \text{pole of order } 2.$$

$$\operatorname{Res}(f, i) = \frac{d}{dz} \frac{e^{iz}}{(z+i)^2}\Bigg|_{z=i} = -\frac{e^{-1}}{2}\, i$$

$$\text{so} \int_{-\infty}^{+\infty} \frac{\cos x}{(1+x^2)^2}\,dx = Re\left(\int_{-\infty}^{+\infty} \frac{e^{ix}}{(1+x^2)^2}\,dx\right)$$

$$= Re\left(2\pi i\left(-\frac{e^{-1}}{2} i\right)\right) = \pi e^{-1}$$

(c). $f(z) = \dfrac{1}{z^4+z^2+1}$

$\deg Q = 4 \geq \deg P + 2 = 0+2$

poles of $f(z)$: $z^4+z^2+1 = 0 \Rightarrow z^2 = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1+\sqrt{3}i}{2} = e^{\frac{2}{3}\pi i}$

or $\dfrac{-1-\sqrt{3}i}{2} = e^{-\frac{2}{3}\pi i}$

$$z = e^{\frac{\pi}{3}i + ik\pi}, \quad k=0,1,$$

$$z = e^{-\frac{\pi}{3}i + ik\pi}, \quad k=0,1$$

$$z_1 = e^{\frac{\pi}{3}i}, \quad z_2 = e^{\frac{4}{3}\pi i}, \quad z_3 = e^{-\frac{\pi}{3}i}, \quad z_4 = e^{\frac{2}{3}\pi i}$$

z_1, z_4 are in the half half.

$$Res(f, z_1) = \frac{1}{4z_1^3+2z_1} = \frac{1}{2z_1(2z_1^2+1)} = \frac{1}{2z_1(-1+\sqrt{3}i+1)} = \frac{1}{2\sqrt{3}i}\frac{1}{z_1}$$

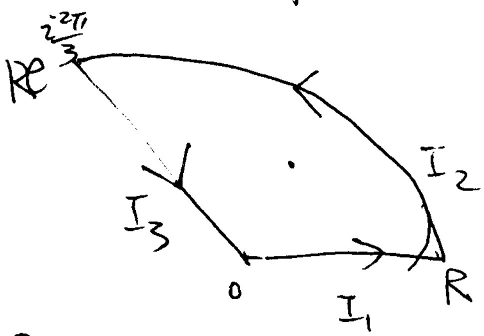
$$= \frac{1}{2\sqrt{3}i} e^{-\frac{\pi}{3}i} = \frac{1}{2\sqrt{3}i}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$Res(f, z_2) = \frac{1}{4z_4^3+2z_4} = \frac{1}{2z_4(-1-\sqrt{3}i+1)} = -\frac{1}{2\sqrt{3}i}\frac{1}{z_4} = -\frac{1}{2\sqrt{3}i}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

So $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^4+x^2+1} = 2\pi i \left(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2) \right)$

$$= 2\pi i \left(\frac{1}{2\sqrt{3}\, i} \right) = \frac{\pi}{\sqrt{3}}$$

(d). Use the loop.



$$\int_{I_1} + \int_{I_2} + \int_{I_3} = 2\pi i \operatorname{Res}\left(\frac{1}{z^3+1},\ e^{i\frac{\pi}{3}} \right)$$

$$\int_{I_1} = \int_0^R \frac{1}{x^3+1}\, dx, \quad \int_{I_3} \underset{\substack{z=\rho e^{i\frac{2\pi}{3}} \\ dz = d\rho\, e^{i\frac{2\pi}{3}}}}{=} e^{i\frac{2\pi}{3}} \int_R^0 \frac{1}{z^3+1}\, d\rho$$

$$= -e^{i\frac{2\pi}{3}} \int_0^R \frac{1}{1+\rho^3}\, d\rho$$

$$= -e^{i\frac{2\pi}{3}} I_1$$

$$\left| \int_{I_2} f(z)\, dz \right| \le \left| \int_0^{\frac{2\pi}{3}} \frac{R}{R^3-1}\, d\rho \right| \le \frac{C}{R^2} \to 0 \text{ as } R \to +\infty$$

$$\operatorname{Res}\left(\frac{1}{z^3+1},\ e^{i\frac{\pi}{3}} \right) = \frac{1}{3z_1^2} = -\frac{z_1}{3} = -\frac{1}{3} e^{i\frac{\pi}{3}}$$

so

$$(1-e^{i\frac{2\pi}{3}})\int_0^{\infty}\frac{1}{x^3+1}\,dx = 2\pi i\left(-\frac{1}{3}e^{i\frac{\pi}{3}}\right)$$

$$\int_0^{\infty}\frac{1}{x^3+1}\,dx = \frac{2\pi}{3\sqrt{3}}$$

3. $f(z) = \dfrac{e^{2z}}{\cosh(\pi z)}$. It has poles at

$$\cosh(\pi z)=0 \Rightarrow e^{\pi z}+e^{-\pi z}=0 \Rightarrow e^{2\pi z}=-1=e^{\pi i+2k\pi i}$$

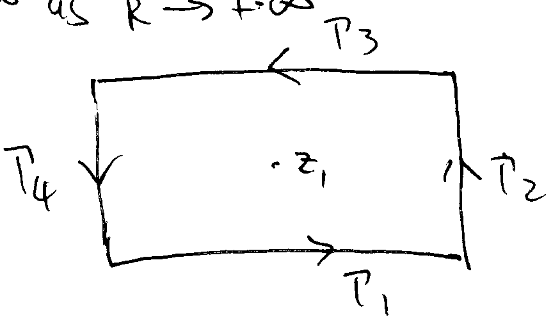
$$2\pi z = \pi i + 2k\pi i$$

$$\Rightarrow z = \frac{i}{2}+ki$$

In side the loop, only $z_1=\frac{i}{2}$ is a pole

$$\mathrm{Res}(f,z_1)=\frac{e^{2z_1}}{\pi\sinh(\pi z_1)} = \frac{2e^{i}}{\pi(e^{\frac{\pi i}{2}}-e^{-\frac{\pi i}{2}})} = \frac{2e^{i-\frac{\pi}{2}i}}{\pi(1-e^{-\pi i})} = \frac{\cancel{2}e^{i-\frac{\pi}{2}i}}{\pi}$$

Now as $R\to+\infty$



on P_2: $z=R+iy$, $0\le y\le 1$

$dz = i\,dy$

$$\left|\frac{e^{2z}}{\cosh\pi z}\right| \le \frac{|e^{2z}|}{|e^{\pi z}|-|e^{-\pi z}|} \le e^{(2-\pi)R}$$

$$\int_{P_2} f(z)\,dz \le Ce^{(2-\pi)R} \to 0.$$

On Γ_4, ~~z~~ $= -R + i y$.

$$\left| \frac{e^{2z}}{\cosh \pi z} \right| \le \frac{|e^{2z}|}{|e^{-\pi z}| - |e^{\pi z}|} \le \frac{e^{-2R}}{e^{\pi R}} \le e^{-(2+\pi)R} \to 0$$

on Γ_3, $z = i + x$, $-R < x < R$

so
$$\frac{e^{2z}}{\cosh(\pi z)} = \frac{2 e^{2i} e^{2x}}{e^{\pi i + \pi x} + e^{-i\pi - \pi x}} = -\frac{e^{2i} e^{2x}}{\cosh(\pi x)}$$

$$\int_{\Gamma_1} f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx$$

$$\int_{\Gamma_3} f(z)\,dz = \cancel{+}\, e^{2i} \int_{-\infty}^{+\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx$$

So
$$\int_{-\infty}^{+\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx = 2\pi i \frac{\cancel{2}\, e^{i - \frac{\pi}{2} i}}{\pi (1 + e^{2i})} = + \frac{\cancel{2}}{\cancel{\cos}\ 1}$$