Solutions to Assignment 2

1. Let $f(z) = \dfrac{1}{z^2 - 2z + 2}$. Then $|f(z)| \le \dfrac{C}{|z|^2}$ for $|z|$ large

We follow Lecture Note 2.

$$\sum_{n=-\infty}^{+\infty} f(n) = - \sum_{j=1}^{S} \operatorname{Res}\left[\pi f(z) \cot(\pi z), \xi_j\right]$$

where ξ_j are poles of $f(z)$: $\quad z^2 - 2z + 2 = 0$

$$\Rightarrow \quad (z-1)^2 = -1 \quad \Rightarrow \quad z_1 = 1 + i, \quad z_2 = 1 - i$$

$$f(z) = \frac{1}{(z - z_1)(z - z_2)}$$

$$\operatorname{Res}\left[\pi f(z) \cot(\pi z); z_1\right] = \pi \cot(\pi z_1) \cdot \frac{1}{2z_1 - 2}$$

$$= \frac{\pi}{2i} \cot(\pi(1+i))$$

$$= -\frac{\pi}{2} i \, \frac{e^{i(\pi(1+i))} + e^{-i(\pi(1+i))}}{e^{i(\pi(1+i))} - e^{-i(\pi(1+i))}} \cdot 2i$$

$$= \pi \cdot \frac{e + e^{-1}}{e^{-1} - e}$$

$$\operatorname{Res}\left[\pi f(z) \cot(\pi z); z_2\right] = \frac{\pi}{-2i} \cot(\pi(1-i)) = \pi \, \frac{e + e^{-1}}{e^{-1} - e}$$

Hence $\displaystyle\sum_{n=-\infty}^{+\infty} f(n) = 2\pi \, \frac{e^2+1}{e^2-1}$.

2. (a) $1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$$w = \log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right), \quad k = 0, \pm 1, \cdots$$

(b) $$w = e^{(1+i)\log(1+i)}$$

$$= e^{(1+i)\ln\sqrt{2} + (1+i)i\left(\frac{\pi}{4}+2k\pi\right)}, \quad k=0, \pm 1, \cdots$$

$$= e^{\ln\sqrt{2} - \frac{\pi}{4} - 2k\pi + i\left(\ln\sqrt{2} + \frac{\pi}{4} + 2k\pi\right)}$$

(c) $$w = \frac{1}{i}\log\left(z + (z^2-1)^{\frac{1}{2}}\right)$$

$z = 2i \quad \cancel{= \sqrt{2}\, e^{i\frac{\pi}{4}}}$

$\cancel{(2i)}\ z^2 - 1 = (2i)^2 - 1 = -5 \Rightarrow \quad (z^2-1)^{\frac{1}{2}} = (-5)^{\frac{1}{2}} = \pm\sqrt{5}\, i$

So $$w = \frac{1}{i}\log\left(2i \pm \sqrt{5}\, i\right)$$

"+" sign $\Rightarrow$ $$w = \frac{1}{i}\log(2+\sqrt{5})i$$

$$= \frac{1}{i}\left[\ln(2+\sqrt{5}) + i\left(\frac{\pi}{2} + 2k\pi\right)\right], \quad k=0, \pm 1, \cdots$$

"−" sign $\Rightarrow$ $$w = \frac{1}{i}\log\left((2-\sqrt{5})i\right)$$

$$= \frac{1}{i}\left(\ln|2-\sqrt{5}| + i\left(\frac{3\pi}{2} + 2k\pi\right)\right), \quad k=0, \pm 1, \cdots$$

3. (a). Branch cut

$$\sqrt{z} = \rho^{\frac{1}{2}} e^{\frac{i}{2}\varphi} \qquad 0<\varphi<2\pi$$

(b) Branch cut

$$\sqrt{z} = \rho^{\frac{1}{2}} e^{\frac{i}{2}\varphi},$$

$$-\frac{\pi}{2}<\varphi<\frac{3\pi}{2}$$

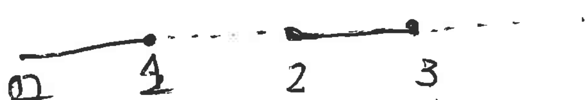
(c) $\sqrt{z(z-1)}$,

$$= \rho_1^{\frac{1}{2}} \rho_2^{\frac{1}{2}} e^{\frac{i}{2}(\varphi_1+\varphi_2)}, \qquad 0<\varphi_1<2\pi, \ 0<\varphi_2<2\pi$$

(d) $\sqrt{z(z-1)(z-2)(z-3)}$

$$= \rho_1^{\frac{1}{2}} \rho_2^{\frac{1}{2}} \rho_3^{\frac{1}{2}} \rho_4^{\frac{1}{2}} e^{\frac{i}{2}(\varphi_1+\varphi_2+\varphi_3+\varphi_4)}$$

$$0<\varphi_1<2\pi, \quad 0<\varphi_2<2\pi, \quad 0<\varphi_3<2\pi, \ 0<\varphi_4<2\pi$$



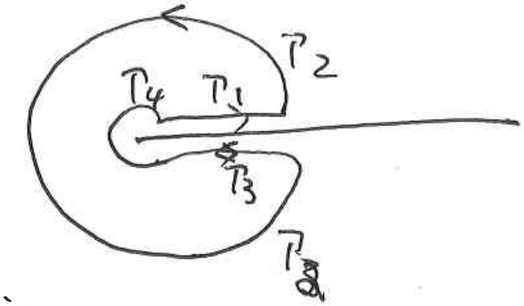
check $(1,2)$: $\varphi_1^+=0$, $\varphi_2^+=0$, $\varphi_3^+=\pi$, $\varphi_4^+=\pi$

$\varphi_1^-=2\pi$, $\varphi_2^-=2\pi$, $\varphi_3^-=\pi$, $\varphi_4^-=\pi$

so $e^{\frac{i}{2}(\varphi_1^++\varphi_2^++\varphi_3^++\varphi_4^+)} = e^{\frac{i}{2}(\varphi_1^-+\varphi_2^-+\varphi_3^-+\varphi_4^-)}$

Similar argument for $(3,+\infty)$

4. (a). Use the contour 1



$$f(z)=z^{-\frac{1}{3}}\frac{1}{1+z^2}$$

On Γ_1, $z=\rho e^{i0}$, $dz=d\rho$

$$f(z)=\rho^{-\frac{1}{3}}\frac{1}{1+\rho^2}, \quad I_1=\int_\varepsilon^R \frac{\rho^{-\frac{1}{3}}}{1+\rho^2}\,d\rho$$

on Γ_3, $z=\rho e^{i\cdot 2\pi}$, $dz=d\rho\, e^{2\pi i}$

$$f(z)=\rho^{-\frac{1}{3}}e^{-\frac{2\pi}{3}i}\frac{1}{1+\rho^2}$$

$$I_3=\int_R^\varepsilon \frac{\rho^{-\frac{1}{3}}e^{-\frac{2\pi}{3}i}}{1+\rho^2}\,d\rho\, e^{2\pi i} = -e^{-\frac{2\pi}{3}i}\int_R^\varepsilon \frac{\rho^{-\frac{1}{3}}}{1+\rho^2}\,d\rho$$

On Γ_2. $|f(z)|\le R^{-\frac{1}{3}}\frac{1}{R^2}$, $\left|\int_{\Gamma_2} f(z)\,dz\right| \le R^{-\frac{1}{3}}\frac{1}{R^2}R \to 0$ as $R\to\infty$

on Γ_4 $|f(z)|\le |\varepsilon|^{-\frac{1}{3}}\frac{1}{1-\varepsilon^2}$, $\left|\int_{\Gamma_4} f(z)\,dz\right| \le \varepsilon^{-\frac{1}{3}}\frac{1}{1-\varepsilon^2}\cdot\varepsilon \to 0$ as $\varepsilon\to 0$

~~f~~ $f(z)$ has poles at $z_1=i$, $z_2=-i$

$$f(z)=\frac{z^{-\frac{1}{3}}}{1+z^2} \qquad \operatorname{Res}(f,i)=\frac{i^{-\frac{1}{3}}}{2i}=\frac{e^{-\frac{\pi}{6}i}}{2i}$$

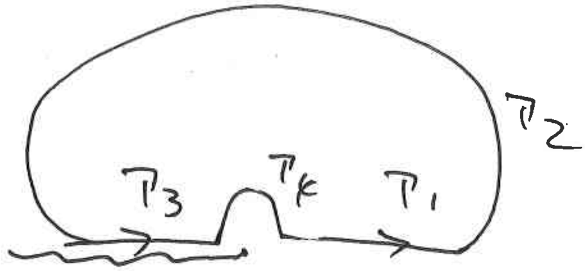
$$\operatorname{Res}(f,-i)=\frac{(-i)^{-\frac{1}{3}}}{-2i}=\frac{e^{-\frac{\pi}{2}i}}{-2i}$$

Thus

$$\left(1-e^{-\frac{2\pi}{3}i}\right)\int_0^{+\infty}\frac{\rho^{-\frac{1}{3}}}{1+\rho^2}\,d\rho = \frac{1}{2i}\left[e^{-\frac{\pi}{6}i}-e^{-\frac{\pi}{2}i}\right]\cdot 2\pi i$$

$$\int_0^{+\infty}\frac{\rho^{-\frac{1}{3}}}{1+\rho^2}\,d\rho = \frac{\pi}{\sqrt{3}}$$

(b). We use the contour 2.



Branch cut for $\log z$ — principal

~~On Γ_1~~,

$$f(z) = \frac{\log z}{z^2+4}$$

On Γ_1, $\quad I_1 = \int_\varepsilon^R \frac{\log x}{x^2+4}\,dx$

on Γ_3, $\quad z = \rho e^{i\pi}$, $\quad \mathrm{Log}\, z = \log \rho + i\pi$

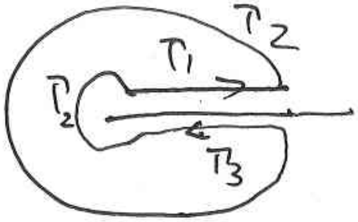
$$I_3 = \int_R^\varepsilon \frac{\log \rho + i\pi}{\rho^2+4}\,d\rho\, e^{i\pi} = \int_\varepsilon^R \frac{\log \rho + i\pi}{\rho^2+4}\,d\rho$$

$$\mathrm{Res}(f, 2i) = \frac{\mathrm{Log}\, 2i}{2.2i} = \frac{\log 2 + i\frac{\pi}{2}}{4i}$$

so $$\int_0^{+\infty} \frac{2\log x + i\pi}{x^2+4}\,dx = 2\pi i \cdot \frac{\log 2 + i\frac{\pi}{2}}{4i} = \frac{\pi}{2}\log 2 + i\frac{\pi^2}{4}$$

so $$\int_0^{+\infty} \frac{\log x}{x^2+4}\,dx = \frac{\pi}{4}\log 2.$$

(c). We use contour 1.



On Γ_1, $z = \rho e^{i0}$, $\sqrt{z}\log z = \rho^{\frac{1}{2}}\log\rho$.

On Γ_3, $z = \rho e^{i2\pi}$ $\sqrt{z}\log z = \rho^{\frac{1}{2}} e^{i\pi}(\log\rho + i(2\pi))$

$$= -\rho^{\frac{1}{2}}(\log\rho + i2\pi)$$

$$I_1 = \int_\varepsilon^R \frac{\rho^{\frac{1}{2}}\log\rho}{\rho^2+1}\,d\rho, \quad I_3 = \int_R^\varepsilon -\frac{\rho^{\frac{1}{2}}(\log\rho + i2\pi)}{\rho^2+1}\,d\rho$$

pole : $\pm i$,

$$\mathrm{Res}(f;i) = \frac{\sqrt{i}\log i}{2i} = \frac{e^{i\frac{\pi}{4}}(i\frac{\pi}{2})}{2i}$$

$$\mathrm{Res}(f;-i) = \frac{\sqrt{-i}\log(-i)}{-2i} = \frac{e^{i\frac{3\pi}{4}}(i\frac{3\pi}{2})}{-2i}$$

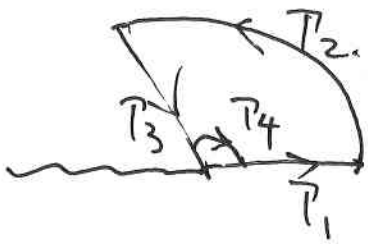
So

$$2\int_0^{+\infty} \frac{\rho^{\frac{1}{2}}\log\rho}{\rho^2+1} = \mathrm{Real}\left[2\pi i \cdot \frac{e^{i\frac{\pi}{4}}\frac{\pi}{2}i}{2i} + 2\pi i \cdot \frac{e^{i\frac{3\pi}{4}}(i\frac{3\pi}{2})}{-2i}\right]$$

$$= \frac{\sqrt{2}}{2}\pi^2$$

$$\int_0^{+\infty} \frac{\rho^{\frac{1}{2}}\log\rho}{\rho^2+1} = \frac{\sqrt{2}}{4}\pi^2$$

(d). We use contour 3.



On Γ_1, $I_1 = \displaystyle\int_0^{+\infty} \frac{\log \rho}{\rho^3+1}\, d\rho$

On Γ_3, $\cancel{z} = \rho e^{\frac{2\pi i}{3}}$, $\quad \operatorname{Log} z = \log \rho + i \cdot \frac{2\pi}{3}$

$$I_3 = \int_R^{\varepsilon} \frac{\log\rho + i\cdot\frac{2\pi}{3}}{\rho^3+1}\, d\rho\ e^{\frac{2\pi i}{3}}$$

$$= -e^{\frac{2\pi i}{3}} \int_{\varepsilon}^{R} \frac{\log\rho + i\cdot\frac{2\pi}{3}}{\rho^3+1}\, d\rho$$

pole: $\quad f(z) = \dfrac{\operatorname{Log} z}{z^3+1}$, $\quad z^3+1=0$, $\quad z = e^{\frac{\pi}{3}i}$

$$\operatorname{Res}(f, e^{\frac{\pi}{3}i}) = \frac{\operatorname{Log} e^{\frac{\pi}{3}i}}{3\cdot e^{\frac{2\pi}{3}i}} = \frac{1}{3} e^{-\frac{2\pi}{3}i} \cdot \left(\frac{\pi}{3} i\right)$$

$$\left(1 - e^{\frac{2\pi i}{3}}\right) \int_0^{+\infty} \frac{\log\rho}{\rho^3+1} - e^{\frac{2\pi}{3}i} \int_0^{\infty} \frac{i\,\frac{2\pi}{3}}{\rho^3+1}\, d\rho$$

$$= 2\pi i \left(\frac{1}{3}\right) e^{-\frac{2\pi}{3}i}\, \frac{\pi}{3} i$$

$$\left(e^{-\frac{2\pi}{3}i} - 1\right) \int_0^{+\infty} \frac{\log\rho}{\rho^3+1} - \int_0^{\infty} \frac{i\,\frac{2\pi}{3}}{\rho^3+1}\, d\rho = 2\pi i \left(\frac{1}{3}\right) e^{-\frac{4\pi}{3}i}\, \frac{\pi}{3} i$$

Take real parts:

$$\frac{1}{2} \int_0^{+\infty} \frac{\log\rho}{\rho^3+1} = -\frac{\pi^2}{9} \quad \Rightarrow \quad \int_0^{\infty} \frac{\log\rho}{\rho^3+1}\, d\rho = -\frac{2\pi^2}{9}$$