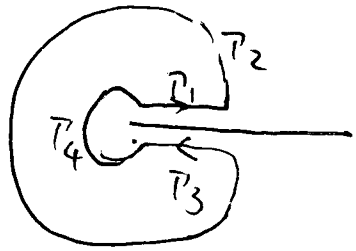
Solutions to Assignment 3

1 (a). Let $f(z) = \operatorname{Log} z \dfrac{z}{(z+1)(z^2+2z+2)}$ with contour



On Γ_1, $z = \rho e^{i0}$, $\operatorname{Log} z = \log \rho$

$$\int_{\Gamma_1} f(z)\,dz = \int_\varepsilon^R \operatorname{Log}\rho \frac{\rho}{(\rho+1)(\rho^2+2\rho+2)}\,d\rho$$

On Γ_3, $z = \rho e^{i \cdot 2\pi}$, $\operatorname{Log} z = \log\rho + i(2\pi)$

$$\int_{\Gamma_3} f(z)\,dz = \int_R^\varepsilon (\operatorname{Log}\rho + i2\pi) \frac{\rho}{(\rho+1)(\rho^2+2\rho+2)}\,d\rho$$

$$= -\int_\varepsilon^R (\log\rho + i2\pi) \frac{\rho}{(\rho+1)(\rho^2+2\rho+2)}\,d\rho$$

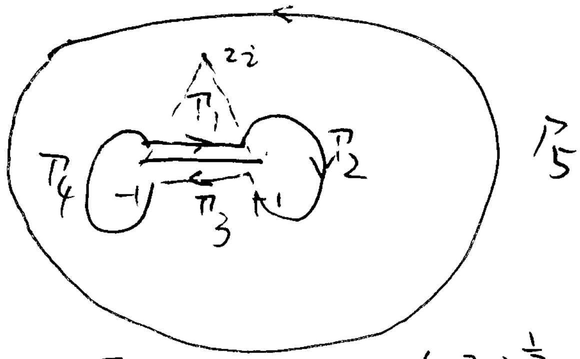
On Γ_2, $|f(z)| \le c(\log R) \cdot \frac{1}{R^2}$, $\int_{\Gamma_2} |f(z)||dz| \le C \log R \frac{1}{R^2} R \to 0$ as $R \to \infty$

On Γ_4, $|f(z)| \le c\,|\log\varepsilon| \frac{\varepsilon}{(1-\varepsilon)}$, $\int_{\Gamma_4} |f(z)||dz| \le C \varepsilon^2 |\log\varepsilon| \to 0$ as $\varepsilon \to 0$.

Residues: $z+1=0$, $z^2+2z+2 = (z+1)^2+1 = 0$

$z_1 = -1$, $z_2 = -1+i$, $z_3 = -1-i$

(b).



$$f(z) = \frac{\sqrt{1-z^2}}{z^2+4} = \frac{\sqrt{-1}\,(z^2-1)^{\frac{1}{2}}}{z^2+4} = \frac{i\,(z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}}{z^2+4}$$

$$= \frac{i\, r_1^{\frac{1}{2}} r_2^{\frac{1}{2}} e^{i(\frac{\varphi_1}{2}+\frac{\varphi_2}{2})}}{z^2+4} \qquad 0<\varphi_1<2\pi,\ 0<\varphi_2<2\pi$$

on Γ_1: $z = \rho e^{i0}$, $\varphi_1 = \pi$, $\varphi_2 = 0$

$$\int_{\Gamma_1} f(z)\,dz = \int_{-1+\varepsilon}^{1-\varepsilon} \frac{i\,|z-1|^{\frac{1}{2}}|z+1|^{\frac{1}{2}} e^{i\frac{\pi}{2}}}{\rho^2+4}\,d\rho = \int_{-1+\varepsilon}^{1-\varepsilon} (-1)\frac{|\rho^2-1|^{\frac{1}{2}}}{\rho^2+4}\,d\rho$$

on Γ_3: $z = \rho e^{i\cdot 0}$, $\varphi_1 = \pi$, $\varphi_2 = 2\pi$

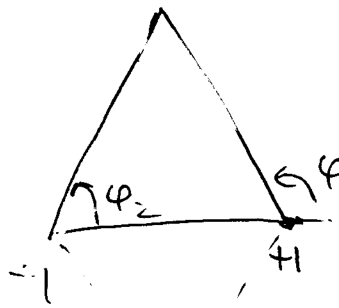
$$\int_{\Gamma_3} f(z)\,dz = \int_{1-\varepsilon}^{-1+\varepsilon} \frac{i\,|z-1|^{\frac{1}{2}}|z+1|^{\frac{1}{2}} e^{i\frac{3\pi}{2}}}{\rho^2+4}\,d\rho = \int_{-1+\varepsilon}^{1-\varepsilon} (-1)\frac{|\rho^2-1|^{\frac{1}{2}}}{\rho^2+4}\,d\rho$$

On Γ_5: $z = Re^{i\varphi}$, $f(z) \sim \frac{iz}{z^2} \sim \frac{i}{z}$

$$\int_{\Gamma_5} f(z)\,dz = \int_0^{2\pi} \frac{i}{z}\, iz\,d\varphi = -2\pi$$

On Γ_2 and Γ_4, the integral $\to 0$.

$$\mathrm{Res}(f, 2i) = \frac{i\,(2i-1)^{\frac{1}{2}}(2i+1)^{\frac{1}{2}}}{2\;2i} = \frac{1}{4}\cdot|2i-1|^{\frac{1}{2}}\,|2i+1|^{\frac{1}{2}}\,e^{i\frac{1}{2}(\varphi_1+\varphi_2)}$$



$\varphi_1 + \varphi_2 = \pi.$

$$= \frac{1}{4}\sqrt{5}\cdot e^{i\frac{\pi}{2}} = \frac{1}{4}\sqrt{5}\,i$$

$$\mathrm{Res}(f, 2i) = \frac{i\,(-2i-1)^{\frac{1}{2}}(-2i+1)^{\frac{1}{2}}}{2\cdot(-2i)} = -\frac{1}{4}\sqrt{5}\;e^{i\frac{3\pi}{2}}$$

$$= \frac{1}{4}\sqrt{5}\;i$$

so

$$(-2)\int_{-1}^{1}\frac{(1-x^2)^{\frac{1}{2}}}{x^2+4}\,d\rho = 2\pi i\left(\frac{1}{4}\sqrt{5}\,i + \frac{1}{4}\sqrt{5}\,i\right) - 2\pi$$

$$= -\sqrt{5}\,\pi - 2\pi$$

$$\int_{-1}^{1}\frac{(1-x^2)^{\frac{1}{2}}}{x^2+4}\,dx = \frac{\sqrt{5}}{2}\pi + \pi$$

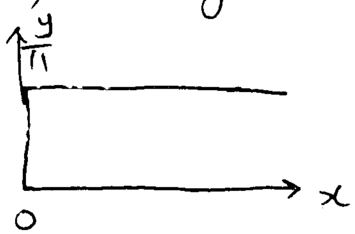
2. $\omega = e^{z}$

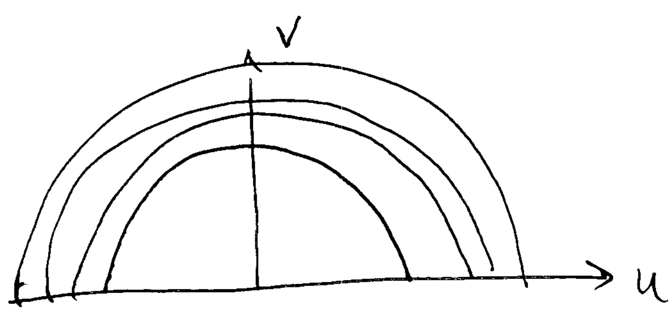
$u + iv = e^{x} e^{iy}$

$u = e^{x} \cos y, \quad v = e^{x} \sin y. \qquad u^2 + v^2 = e^{2x}$

(a) $\mathrm{Re}\, z > 0, \quad 0 < \mathrm{Im}\, z < \pi$

$x > 0, \quad 0 < y < \pi, \qquad u^2 + v^2 = e^{2x} > 1, \quad v > 0$



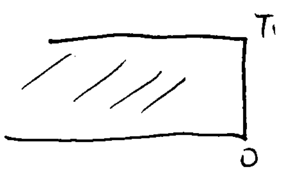


is mapped to

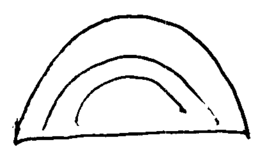
$\{(u,v) \mid u^2 + v^2 > 1, v > 0\}$

(b). $\mathrm{Re}\, z < 0, \quad 0 < \mathrm{Im}\, z < \pi$

$x < 0, \quad 0 < y < \pi \quad \rightarrow \quad u^2 + v^2 = e^{2x} < 1, \quad v > 0$



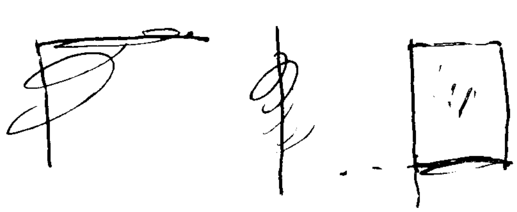
$\longrightarrow$



$\{(u,v) \mid u^2 + v^2 < 1, \ v > 0\}$

(c) $1 < \mathrm{Re}\, z < 2, \ 0 < \mathrm{Im}\, z < \pi$

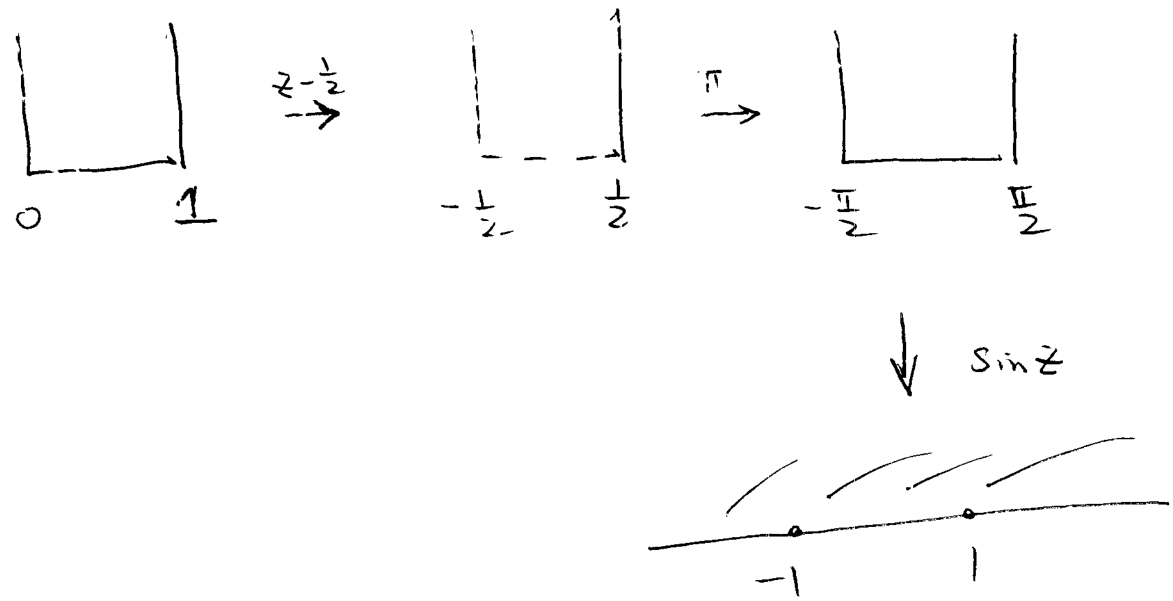
$u^2 + v^2 = e^{2x} \in (e, e^2), \quad v = e^{x} \sin y > 0.$



$\rightarrow$

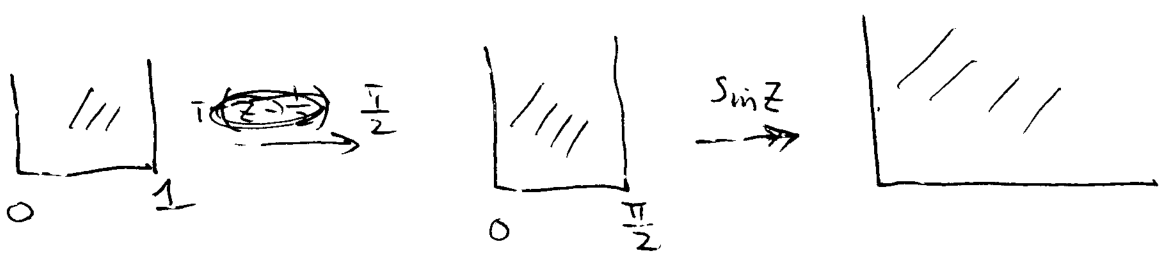
$\{(u,v) \mid u^2 + v^2 \in (e, e^2), \ v > 0\}$

3 (a).



$$w = \sin\left(\pi\left(z-\frac{1}{2}\right)\right)$$

(b)



$$w = \sin\left(\frac{\pi}{2} z\right).$$

4. (a). $\phi = A + B_1 \operatorname{Arg}(z+1) + B_2 \operatorname{Arg}(z) + B_3 \operatorname{Arg}(z-1)$

$A + B_1 \cdot 0 + B_2 \cdot 0 + B_3 \cdot 0 = -2 \quad \Rightarrow A = -2$

$A + B_1 \cdot 0 + B_2 \cdot 0 + B_3 \cdot \pi = 2 \quad \Rightarrow B_3 = -\frac{4}{\pi}$

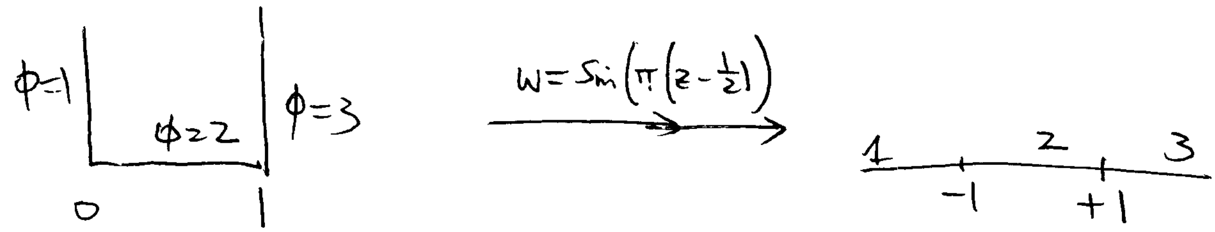
$A + B_1 \cdot 0 + B_2 \cdot \pi + B_3 \cdot \pi = 0 \quad \Rightarrow B_2 = -\frac{2}{\pi}$

$A + B_1 \cdot \pi + B_2 \cdot \pi + B_3 \cdot \pi = 1 \quad \Rightarrow B_1 = \frac{1}{\pi}$

So

$$\phi = -2\ \cancel{\tfrac{3}{\pi}}\tfrac{1}{\pi}\,\text{Arctan}\,\frac{y}{x+1} - \frac{2}{\pi}\,\text{Arctan}\,\frac{y}{x} - \frac{4}{\pi}\,\text{Arctan}\,\frac{y}{x-1}$$

(b).



For w, $\quad \Phi = A + B_1\,\text{Arg}(z+1) + B_2\,\text{Arg}(z-1)$

$$A + B_1\cdot 0 + B_2\cdot 0 = 3 \quad\Rightarrow\quad A = 3$$

$$A + B_1\cdot 0 + B_2\cdot\pi = 2 \quad\Rightarrow\quad B_2 = -\frac{1}{\pi}$$

$$A + B_1\cdot\pi + B_2\cdot\pi = 1 \quad\Rightarrow\quad B_1 = -\frac{1}{\pi}$$

$$\Phi = 3 - \frac{1}{\pi}\,\text{Arctan}\left(\frac{v}{u+1}\right) - \frac{1}{\pi}\,\text{Arctan}\left(\frac{v}{u-1}\right)$$

$$u + iv = \sin\left(\pi\left(z-\tfrac{1}{2}\right)\right) = -\cos\pi z = -\cosh\pi y\cos\pi x + i\sinh\pi y\sin\pi x$$

$$u = -\cosh\pi y\cos\pi x, \quad v = \sinh\pi y\sin\pi x$$

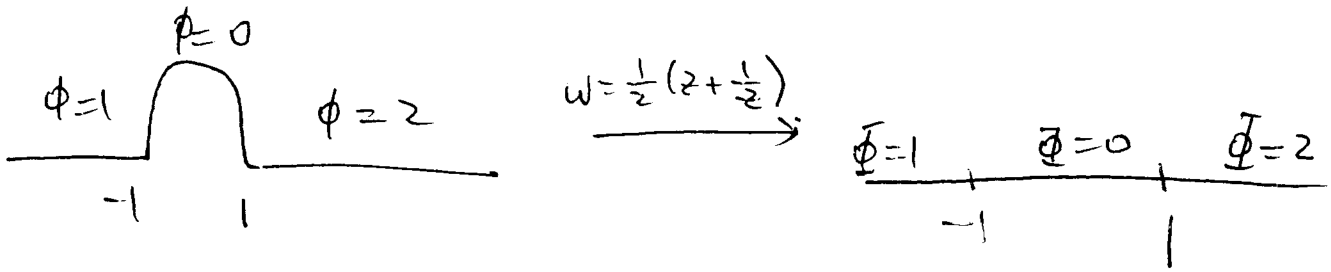
$$\phi = 3 - \frac{1}{\pi}\,\text{Arctan}\left(\frac{\sinh\pi y\sin\pi x}{-\cosh\pi y\cos\pi x + 1}\right) - \frac{1}{\pi}\,\text{Arctan}\left(\frac{\sinh\pi y\sin\pi x}{-\cosh\pi y\cos\pi x - 1}\right)$$

(c). Try $\Phi = A + Bx$

$A = 0, \quad B = 3.$

solution is $\phi = 3x.$

(d).



$$\Phi = A + B_1 \operatorname{Arg}(w+1) + B_2 \operatorname{Arg}(w-1)$$

$$A + B_1 \cdot 0 + B_2 \cdot 0 = 2 \quad \Rightarrow A = 2$$

$$A + B_1 \cdot 0 + B_2 \cdot \pi = 0 \quad \Rightarrow B_2 = -\frac{2}{\pi}$$

$$A + B_1 \cdot \pi + B_2 \cdot \pi = 1 \quad \Rightarrow B_1 = \frac{1}{\pi}$$

$$\Phi = 2 + \frac{4}{\pi} \operatorname{Arctan}\left(\frac{v}{u+1}\right) - \frac{2}{\pi} \operatorname{Arctan}\left(\frac{v}{u-1}\right).$$

$$u + iv = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right) \cancel{e^{i\varphi}} \cos\varphi + \frac{i}{2}\left(\rho - \frac{1}{\rho} \sin\varphi\right)$$

$$u = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\varphi \, ; \quad v = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\varphi$$

$$u = \frac{1}{2}\left(x + \frac{x}{x^2+y^2}\right), \quad v = \frac{1}{2}\left(y - \frac{y}{x^2+y^2}\right).$$

$$\phi = 2 + \frac{1}{\pi} \operatorname{Arctan}\left(\frac{y(x^2+y^2) - y}{x(x^2+y^2) + x + x^2 + y^2}\right) - \frac{2}{\pi} \operatorname{Arctan}\left(\frac{y(x^2+y^2) - y}{x(x^2+y^2) + x - x^2 - y^2}\right)$$