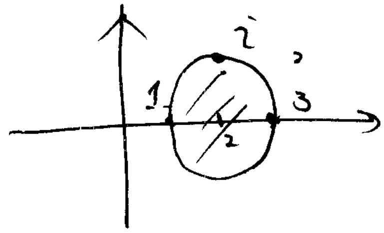
Solutions to Assignment 4. MATH 301-201

1 (a)



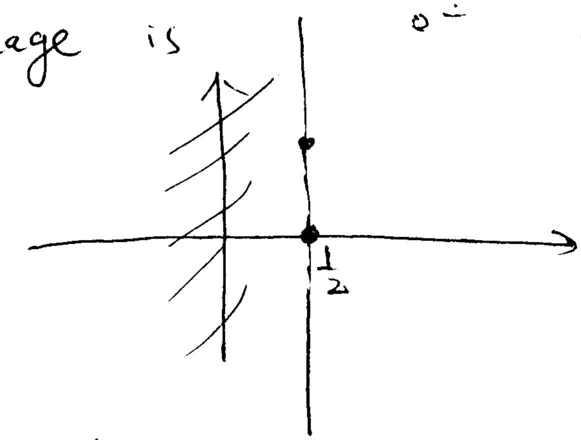
$$w = \frac{z-2}{z-1}$$

1 becomes pole so the image of the circle is a line.

$z = 3 \implies w = \frac{1}{2}$

$z = i+2 \implies w = \cancel{\frac{i-2}{i-1}} \cancel{\frac{(i-2)(i+1)}{(i-1)(i+1)}} \frac{i}{1+i} = \frac{i}{2}(1-i) = \frac{1}{2} + \frac{i}{2}$

So the image is



circle $\to$ line $u = \frac{1}{2}$

interior $\to$ $u < \frac{1}{2}$

(b) $w = \dfrac{z-4}{z-3}$

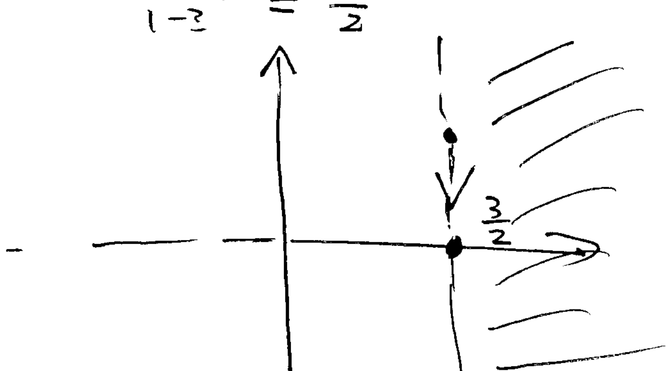
3 becomes pole so the image of circle is a line

$z = 2+i \implies w = \dfrac{-2+i}{-1+i} = 1 + \dfrac{1}{1-i} = 1 + \dfrac{1+i}{2} = \dfrac{3}{2} + \dfrac{i}{2}$

$z = 1 \implies w = \dfrac{1-4}{1-3} = \dfrac{3}{2}$

circle $\Rightarrow u = \frac{3}{2}$

interior $\Rightarrow u > \frac{3}{2}$



2. (a)

z	w
0	0
1	i
∞	∞

$$w = \frac{az+b}{d}$$

$0 \to 0 \Rightarrow b = 0$

$1 \to i \Rightarrow a \cdot 1 = i d \quad \frac{a}{d} = i$

$$w = iz$$

(b).

z	w
$\cancel{0}$	$-i$
1	∞
∞	1

$$w = \frac{az+b}{z-1}$$

$w(\infty)=1 \Rightarrow a=1, \quad w(0) = -i \Rightarrow \frac{b}{-1} = -i, \; b = i$

So $w = \dfrac{z+i}{z-1}$

(c)

z	w
0	1
1	i
∞	∞

$$w = az+b$$

$w(0) = 1 \Rightarrow b = 1, \quad w(1) = i \Rightarrow a+b = i$

$a = i-1$

$$w = (i-1)z + 1$$

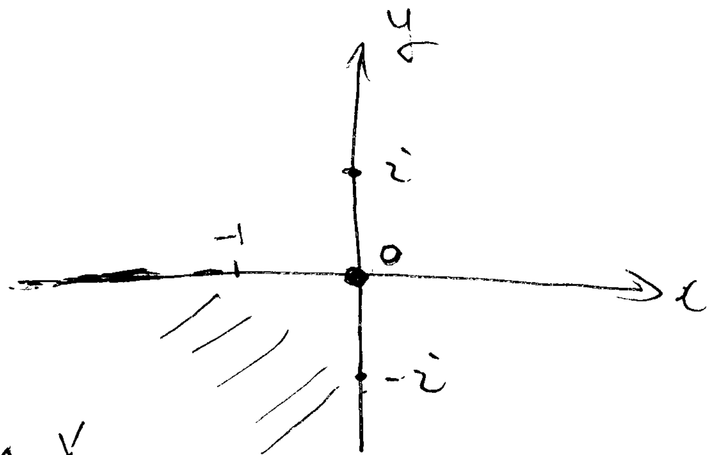
3.

$$w = 1 + \frac{2i}{z - i}$$

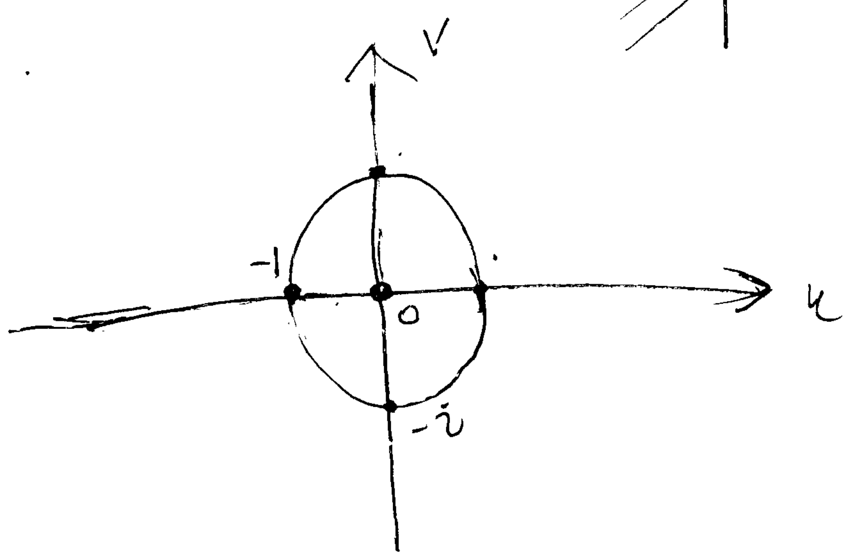
line $x=0$ pass through the pole so it becomes a line

$-i \to 0$

$0 \to -1$



$x=0$ becomes $u=0$.



line $y=0$ doesnot pass through the pole so it becomes a circle.

$-i \to 0$

$i \to \infty$

$-i, i$ are symmetric points. so the circle is centered at the origin. with radius

$$R = |w| = \left|\frac{0+i}{0-i}\right| = 1$$

Now $w(-1) = \dfrac{-1+i}{-1-i} = \dfrac{1-i}{1+i} = -i$

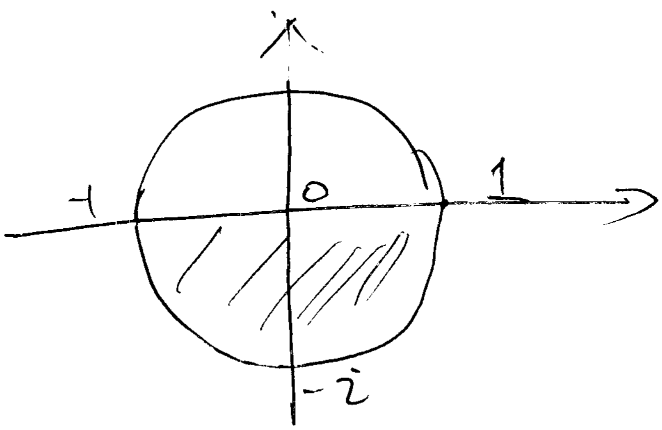
When one walks frome $0, -1, \infty$ on x-axis

we get $-1, -i, 1$

If one walks from $0, -i, \infty$ on y-axis

we get $-1, 0, 1$

Hence the image of the third quadrant is



$\{x<0, y<0\} \longrightarrow \{(u,v) \mid u^2+v^2<1, v<0\}$

4(a) $\phi = A + B \operatorname{Log} r$

$r = 1,\ \phi = 2 \Rightarrow A = 2$

$r = 3,\ \phi = 1 \Rightarrow A + B \operatorname{Log} 3 = 1$

$$B = \frac{-2}{\log 3}$$

So $\phi = 2 - \frac{2}{\log 3} \operatorname{Log} r$.

(b). We need to find two symmetric points z_1 and z_2

It is clear that z_1, z_2 can be real

$$\left.\begin{aligned} (z_2 - 1)(z_1 - 1) &= 1 \\ z_2 z_1 &= 9 \end{aligned}\right\} \qquad \begin{aligned} \left(\frac{9}{z_1} - 1\right)(z_1 - 1) &= 1 \\ (9 - z_1)(z_1 - 1) &= z_1 \end{aligned}$$

$$-9 - z_1^2 + 10 z_1 = z_1$$

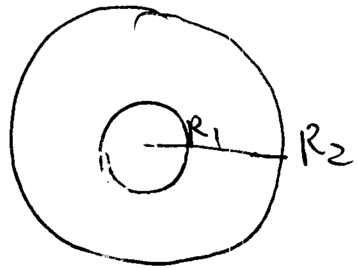
$$z_1^2 - 9 z_1 + 9 = 0$$

$$z_1 = \frac{9 - 3\sqrt{5}}{2}, \quad z_2 = \frac{9 + 3\sqrt{5}}{2}$$

So $w = \dfrac{z - z_1}{z - z_2}$ New

$$R_1 = |w| = \left| \frac{0 - z_1}{0 - z_2} \right| = \frac{9 - 3\sqrt{5}}{9 + 3\sqrt{5}}$$

$$R_2 = |w| = \left| \frac{3 - z_1}{3 - z_2} \right| = \frac{3 + 3\sqrt{5}}{3\sqrt{5} - 3}$$



$$\Phi = A + B \operatorname{Log} \rho$$

$$A + B \operatorname{Log} R_1 = 2$$

$$A + B \operatorname{Log} R_2 = 1$$

$$B = \frac{1}{\operatorname{Log} R_1 - \operatorname{Log} R_2}, \qquad A = 1 - \frac{\operatorname{Log} R_2}{\operatorname{Log} R_1 - \operatorname{Log} R_2} = \frac{\operatorname{Log} R_1 - 2\operatorname{Log} R_2}{\operatorname{Log} R_1 - \operatorname{Log} R_2}$$

So the solution is

$$\phi = \Phi = \frac{\operatorname{Log} R_1 - 2\operatorname{Log} R_2}{\operatorname{Log} R_1 - \operatorname{Log} R_2} + \frac{1}{\operatorname{Log} R_1 - \operatorname{Log} R_2} \operatorname{Log} \rho$$

where $\rho = \sqrt{u^2 + v^2}$ and

$$u + iv = \frac{x + iy - z_1}{x + iy - z_2} = \frac{(x+iy-z_1)(x-iy-z_2)}{(x-z_2)^2 + y^2}$$

$$= \frac{x^2+y^2+z_1z_2 - (z_1+z_2)x + (z_1-z_2)iy}{(x-z_2)^2+y^2}$$

$$u = \frac{x^2+y^2+z_1z_2-(z_1+z_2)x}{(x-z_2)^2+y^2}, \qquad v = \frac{(z_1-z_2)y}{(x-z_2)^2+y^2}$$

(3.) Let $w = \beta \dfrac{z-1}{z-3}$

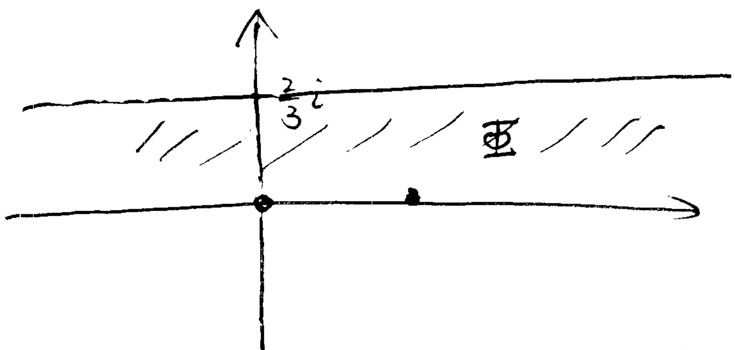
w maps

$1 \to 0$

$3 \to \infty$

$$w(2+i) = \beta \frac{1+i}{-1+i} = \beta \frac{-(1+i)^2}{2} = -\beta i = 1, \quad \beta = i$$

So the inner circle becomes line x-axis.



$w = i \dfrac{z-1}{z-3}$ $\qquad w(\cancel{3}3) = i \dfrac{-4}{-6} = \dfrac{2}{3} i$

$$w(3i) = i \frac{3i-1}{3(i-1)} = i \frac{(3i-1)(i+1)}{3(-2)} = -i \frac{-4+2i}{6} = \frac{1}{3} + \frac{2}{3} i$$

$\Phi = A + Bv$

$v = 0, \; \phi = 2 \Rightarrow A = 2$

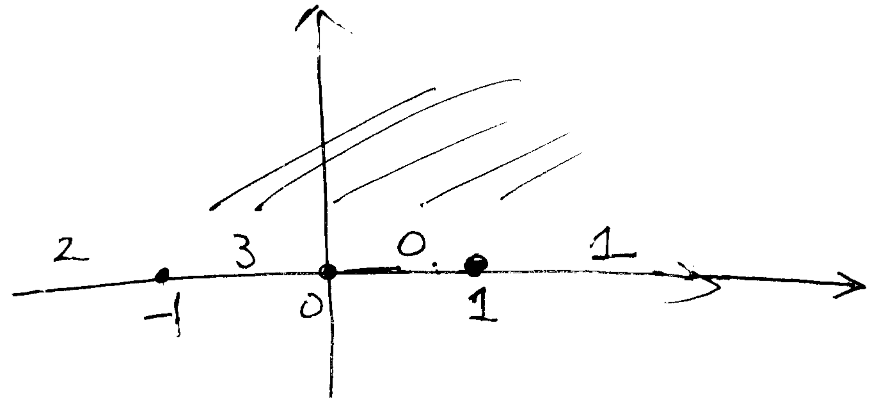
$v = \frac{2}{3}, \; \phi = 1 \Rightarrow 2 + B \cdot \frac{2}{3} = 1 \Rightarrow B = -\frac{3}{2}$

$\Phi = 2 - \frac{3}{2} v$

where $\quad u + iv = i \dfrac{x+iy-1}{x+iy-3} = i \dfrac{(x-1+iy)(x-3-iy)}{(x-3)^2+y^2}$

$$v = \frac{(x-1)(x-3)+y^2}{(x-3)^2+y^2}, \quad \phi = 2 - \frac{3}{2} \cdot \frac{(x-1)(x-3)+y^2}{(x-3)^2+y^2}$$

(d) we first map it to the upper half space



$1 \to 0$

$i \to 1$

$-1 \to \infty$

$$w = (-i)\frac{z-1}{z+1}$$

$$w(-i) = (-i)\frac{-i-1}{-i+1} = (-i)\cdot\frac{-(i+1)}{1-i} = i\,\frac{(1+i)^2}{2} = -1$$

$$\Phi = A + B_1 \operatorname{Arg}(w+1) + B_2 \operatorname{Arg}(w) + B_3 \operatorname{Arg}(w-1)$$

$$A + B_1\cdot 0 + B_2\cdot 0 + B_3\cdot 0 = 1$$

$$A + B_1\cdot 0 + B_2\cdot 0 + B_3\pi = 0$$

$$A + B_1\cdot 0 + B_2\cdot\pi + B_3\pi = 3$$

$$A + B_1\cdot\pi + B_2\pi + B_3\pi = 2$$

$A = 1$, $B_3 = -\frac{1}{\pi}$, $B_2 = \frac{3}{\pi}$, $B_1 = -\frac{1}{\pi}$

$$\Phi = 1 - \frac{1}{\pi}\operatorname{Arg}(w+1) + \frac{3}{\pi}\operatorname{Arg} w - \frac{1}{\pi}\operatorname{Arg}(w-1)$$

where $w = +i\,\frac{1-z}{1+z} = i\,\frac{(1-x)-iy}{(x+1)+iy} = \frac{2y + i(1-x^2-y^2)}{(x+1)^2+y^2}$