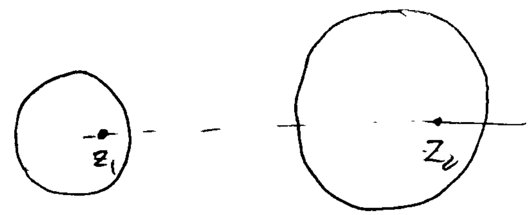
Solutions to Assignment 5 MAT301-201

1. We find symmetric points, z_1 in the first circle, z_2 in the second circle



We may assume that z_1, and z_2 are real, by symmetry.

Hence

$$(z_1+2)(z_2+2)=1 \quad \Rightarrow \quad z_2+2 = \frac{1}{z_1+2}$$

$$(z_1-3)(z_2-3)=4 \quad \Rightarrow \quad z_2-3 = \frac{4}{z_1-3}.$$

$$5+\frac{4}{z_1-3} = \frac{1}{z_1+2} \quad \Rightarrow \quad 5z_1^2-2z_1-19=0$$

$$z_1 = \frac{2 \pm \sqrt{384}}{10} = \frac{1 \pm \sqrt{96}}{5} = \frac{1-4\sqrt{6}}{5} \quad (\text{since } |z_1+2|<1)$$

$$z_2 = \frac{1+4\sqrt{6}}{5}$$

so we may take the mobius transform

$$w = \frac{w-z_1}{w-z_2}$$

The ~~first~~ circle becomes an inner circle with radius

$$R_1 = \left|\frac{-1-z_1}{-1-z_2}\right| = \frac{|3-2\sqrt{6}|}{|3+2\sqrt{6}|} = \frac{2\sqrt{6}-3}{2\sqrt{6}+3}$$

The second circle becomes the outer circle with radius

$$R_2 = \frac{|1-z_1|}{|1-z_2|} = \frac{\sqrt{6}+1}{\sqrt{6}-1}$$

We solve the Laplace Problem

$$\Phi = 1 \quad \text{on } \{|w| = R_1\}$$
$$\Phi = 2 \quad \text{on } \{|w| = R_2\}$$

$$\Phi = A + B \operatorname{Log}|w|$$

$$\left.\begin{aligned} 1 &= A + B \log R_1 \\ 2 &= A + B \operatorname{Log} R_2 \end{aligned}\right\} \Rightarrow B = \frac{1}{\operatorname{Log} R_2 - \operatorname{Log} R_1}$$

$$A = \frac{\operatorname{Log} R_2 - 2 \log R_1}{\operatorname{Log} R_2 - \operatorname{Log} R_1}$$

$$\Phi = \frac{\operatorname{Log} R_2 - 2\log R_1}{\operatorname{Log} R_2 - \operatorname{Log} R_1} + \frac{1}{\operatorname{Log} R_2 - \operatorname{Log} R_1} \operatorname{Log} \sqrt{u^2 + v^2}$$

where.

$$u + iv = \frac{x + iy - z_1}{x + iy - z_2} = \frac{(x - z_1)(x - z_2) + y^2 - iy(z_1 - z_2)}{(x - z_2)^2 + y^2}$$

$$u = \frac{x^2 - (z_1 + z_2)x + z_1 z_2 + y^2}{(x - z_2)^2 + y^2}, \quad v = \frac{-y(z_1 - z_2)}{(x - z_2)^2 + y^2}$$

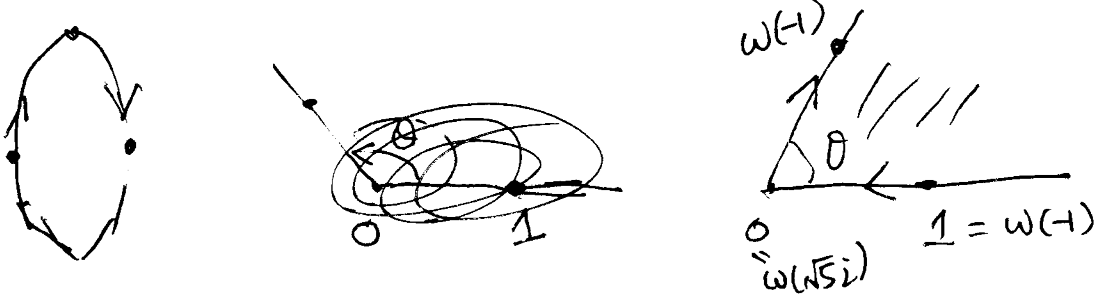
2 (a) $\phi = A + B \operatorname{Arg}(z+1)$.

on $y = 0$, $\operatorname{Arg}(z+1) = 0$, $\quad A = 1$

or $x - y = -1$, $\operatorname{Arg}(z+1) = \frac{\pi}{4}$, $\quad 1 + B \cdot \frac{\pi}{4} = 2 \Rightarrow B = \frac{4}{\pi}$

$$\phi = 1 + \frac{4}{\pi} \operatorname{Arg}(z+1) = 1 + \frac{4}{\pi} \operatorname{Arctan} \frac{y}{x+1}$$

(b) We will map the shaded region into a sector



The two intersection points are $z_1 = \sqrt{5}\,i$ and $z_2 = -\sqrt{5}\,i$

Let $\quad w = \beta \dfrac{z - \sqrt{5}\,i}{z + \sqrt{5}\,i} \quad$ so that $\sqrt{5}\,i \to 0, \ -\sqrt{5}\,i \to \infty$

We map the left hand section into the real axis so that

$$\beta \frac{1-\sqrt{5}\,i}{1+\sqrt{3}\,i} = -1 \quad \Rightarrow \quad \beta = -\frac{1+\sqrt{5}\,i}{1-\sqrt{5}\,i}$$

The other section is mapped to a line

$$w(1) = -\frac{1+\sqrt{5}\,i}{1-\sqrt{5}\,i} \, \frac{1-\sqrt{5}\,i}{1+\sqrt{5}\,i} = \frac{1}{9} + \frac{4\sqrt{5}}{9} i$$

So the angle

$$\tan\theta = \frac{y}{x} = \frac{4\sqrt{5}}{1} = 4\sqrt{5}$$

(Notice the direction). Hence

$$\Phi = A + B \,\mathrm{Arctan}\, \frac{u}{v} \qquad \begin{aligned} &A = 2 \\ &A + B\,\mathrm{Arctan}\, 4\sqrt{5} = 1 \end{aligned} \Bigg\} \Rightarrow B = -\frac{1}{\mathrm{Arctan}\, 4\sqrt{5}}$$

$$\Phi = \cancel{A} - \frac{1}{\mathrm{arctan}\, 4\sqrt{5}} \,\mathrm{Arctan}\, \frac{u}{v}$$

where $\quad u + iv = -\dfrac{1+\sqrt{5}\,i}{1-\sqrt{5}\,i} \, \dfrac{x+iy-\sqrt{5}\,i}{x+iy+\sqrt{5}\,i}$

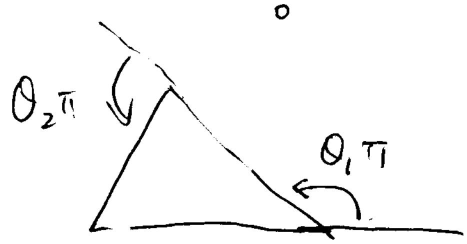
3. We want to find a Schwarz – Christoffel map so that

$-1 \longrightarrow w_1$

$1 \rightarrow i$

$\infty \rightarrow -1$.

Hence
$$w = A \int_0^z (z+1)^{-\theta_1} (z-1)^{-\theta_2} \, dz + B .$$



$$\theta_1 = \frac{3}{4}, \quad \theta_2 = \frac{1}{2}$$

$$w(-1) = 1 \quad \Rightarrow \quad A \int_0^{-1} (1+z)^{-\frac{3}{4}} (z-1)^{-\frac{1}{2}} \, dz + B = 1$$

$$w(1) = i \quad \Rightarrow \quad A \int_0^{1} (1+z)^{-\frac{3}{4}} (z-1)^{-\frac{1}{2}} \, dz + B = i$$

Now
$$\int_0^{-1} (1+z)^{-\frac{3}{4}} (z-1)^{-\frac{1}{2}} \, dz = -\int_0^1 (1-z)^{-\frac{3}{4}} (-z-1)^{-\frac{1}{2}} \, dz$$
$$= -(-1)^{-\frac{1}{2}} \int_0^1 (1-z)^{-\frac{3}{4}} (1+z)^{-\frac{1}{2}} \, dz$$
$$= -e^{-\frac{\pi}{2} i} \int_0^1 (1-z)^{-\frac{3}{4}} (1+z)^{-\frac{1}{2}} \, dz$$
$$= i \int_0^1 (1-z)^{-\frac{3}{4}} (1+z)^{-\frac{1}{2}} \, dz$$

$$\int_0^1 (1+z)^{-\frac{3}{4}} (z-1)^{-\frac{1}{2}} \, dz = (-1)^{-\frac{1}{2}} \int_0^1 (1+z)^{-\frac{3}{4}} (1-z)^{-\frac{1}{2}} \, dz$$
$$= -i \int_0^1 (1+z)^{-\frac{3}{4}} (1-z)^{-\frac{1}{2}} \, dz$$

Hence

$$A\, i \int_0^1 (1-z)^{-\frac{3}{4}}(1+z)^{-\frac{1}{2}}\,dz \quad + B = 1$$

$$-A\, i \int_0^1 (1+z)^{-\frac{3}{4}}(1-z)^{-\frac{1}{2}}\,dz \quad + B = i$$

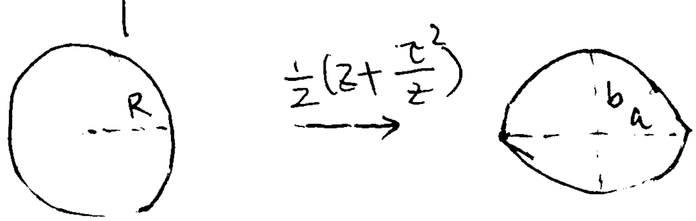
So
$$A = \frac{1-i}{i} \cdot \frac{1}{\int_0^1 (1-z)^{-\frac{3}{4}}(1+z)^{-\frac{1}{2}}dz + \int_0^1 (1+z)^{-\frac{3}{4}}(1-z)^{-\frac{1}{2}}dz}$$

$$= (1-i)\,\frac{1}{\int_{-1}^1 (1-z)^{-\frac{3}{4}}(1+z)^{-\frac{1}{2}}\,dz}$$

$$B = 1 - (1+i)\,\frac{\int_0^1 (1-z)^{-\frac{3}{4}}(1+z)^{-\frac{1}{2}}\,dz}{\int_{-1}^1 (1-z)^{-\frac{3}{4}}(1+z)^{-\frac{1}{2}}\,dz}$$

4. (a) The map
$$\Omega(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$
will do the job.

(b). First we note that the map
$$\Omega(z) = \frac{1}{2}\left(z + \frac{\tau^2}{z}\right) \quad \text{maps}$$

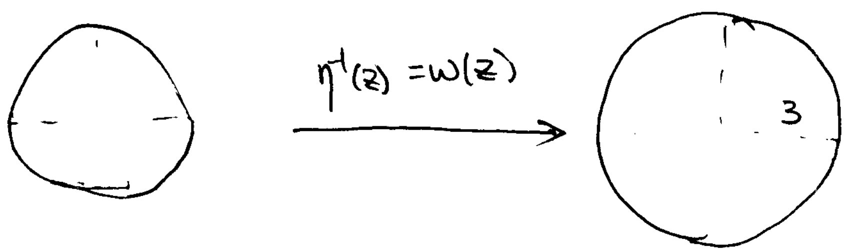


$$a = \frac{1}{2}\left(R + \frac{\tau^2}{R}\right)$$
$$b = \frac{1}{2}\left(R - \frac{\tau^2}{R}\right)$$

So

$$2 = \frac{1}{2}\left(R + \frac{\tau^2}{R}\right)$$
$$1 = \frac{1}{2}\left(R - \frac{\tau^2}{R}\right) \quad \Rightarrow \quad R = 3, \ \tau = \sqrt{3}.$$

The inverse the above map maps ellipse to circle



Let us find the inverse map:

$$\eta = \frac{1}{2}\left(z + \frac{3}{z}\right) \Rightarrow z^2 - 2\eta z + 3 = 0$$
$$\Rightarrow z = \eta \pm \sqrt{\eta^2 - 3}.$$

Since we need $z \approx \eta$, we choose $z = \eta + \sqrt{\eta^2 - 3}$.

So the inverse map $\eta^{-1}(z) = z + \sqrt{z^2 - 3}$

Now ~~the map~~ we solve the problem in w-plane:

$$\Omega = \frac{1}{2}\left(w + \frac{9}{w^2}\right). \qquad w = \eta^{-1}(z) = z + \sqrt{z^2 - 3}$$

Hence $$\Omega = \frac{1}{2}\left(z + \sqrt{z^2-3} + \frac{9}{(z + \sqrt{z^2-3})}\right)$$