Solutions to Assignment 6, MATH301-201

1 (a). We can compute V in two ways.

$$\mathrm{Re}(\Omega(z)) = V_0\left(x + \frac{a^2}{x^2+y^2}x\right) - \frac{\gamma}{2\pi}\,\mathrm{Arctan}\,\frac{y}{x} = \Phi$$

$$V = \nabla\Phi \quad\Rightarrow$$

$$V_1 = \frac{\partial\Phi}{\partial x} = V_0\left(1 + \frac{a^2}{x^2+y^2} - \frac{2a^2x^2}{(x^2+y^2)^2}\right) + \frac{\gamma}{2\pi}\frac{y}{x^2+y^2}$$

$$V_2 = \frac{\partial\Phi}{\partial y} = V_0\left(-\frac{2a^2xy}{(x^2+y^2)^2}\right) - \frac{\gamma}{2\pi}\frac{x}{x^2+y^2}$$

Another way:

$$V = \Omega'(z) = V_0\left(1-\frac{a^2}{z^2}\right) + i\frac{\gamma}{2\pi z} \quad\Rightarrow$$

$V_1 = \cdots,\quad V_2 = \cdots$

(b). $\Omega'(z)=0$ gives stagnation points.

$$z^2-a^2+i\frac{\gamma}{2\pi V_0}z = 0$$

$$z = -\frac{i\gamma}{4\pi V_0} \pm\sqrt{a^2-\frac{\gamma^2}{16\pi^2V_0^2}}$$

Critical γ_c is: $\quad a^2-\frac{\gamma_c^2}{16\pi^2V_0^2} = 0 \quad\Rightarrow\quad \gamma_c = 4\pi a V_0$

where $\gamma<\gamma_c \quad\Rightarrow$

$$z = \frac{-i\gamma}{4\pi V_0} \pm\sqrt{a^2-\frac{\gamma^2}{16\pi^2V_0^2}}$$

where $\gamma>\gamma_c \quad\Rightarrow$

$$z = -i\frac{\gamma}{4\pi V_0} \pm\sqrt{\frac{\gamma^2}{16\pi^2V_0^2}-a^2}\; i$$

where $\gamma < \gamma_c$

$$|z|^2 = a^2 \Rightarrow \text{stagnation points on the body}$$

$\{|z| = a\}$

Note that $\Psi = \text{Constant}$ on $z = ae^{i\varphi}$ since

$$\mathrm{Im}(\Omega) = \mathrm{Im}\left(a v_0 (e^{i\varphi} + e^{-i\varphi}) + \frac{i\gamma}{2\pi}(\log a + i\varphi)\right)$$

$$= \frac{\gamma}{2\pi}\log a.$$

So $\{|z| = a\}$ is the body of the flow.

where $\gamma > \gamma_c$, then

$$z_1 = -\frac{\gamma}{4\pi v_0} + \sqrt{\frac{\gamma^2}{16\pi^2 v_0^2} - a^2}, \quad |z_1| < a$$

$$z_2 = -\frac{\gamma}{4\pi v_0} - \sqrt{\frac{\gamma^2}{16\pi^2 v_0^2} - a^2}, \quad |z_2| > a.$$

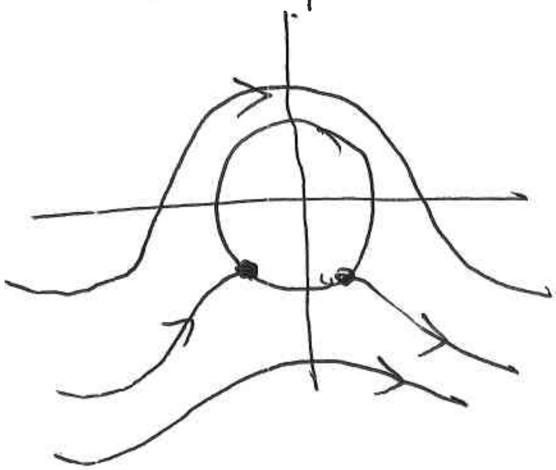
The stagnation point is z_2.

(c). Let $z = re^{i\varphi}$. Then

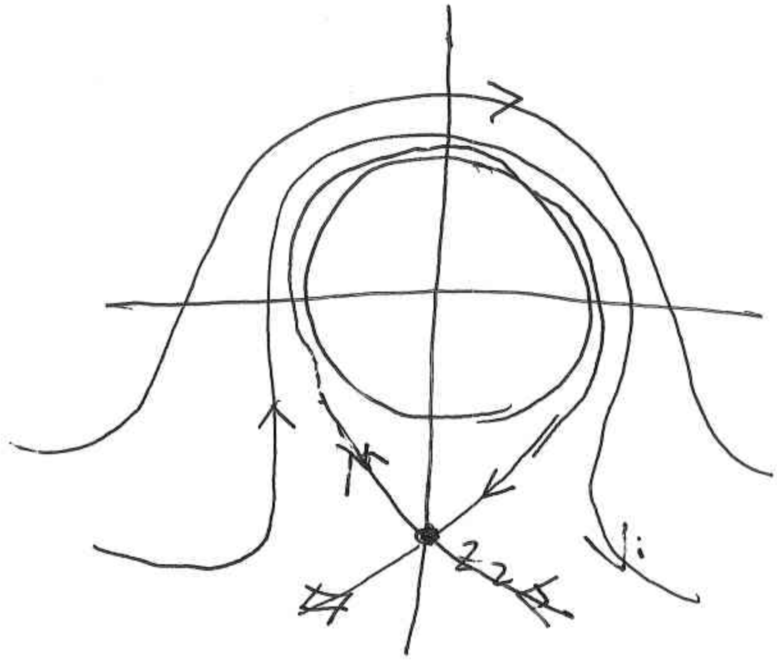
$$\mathrm{Im}(\Omega) = v_0\left(r - \frac{a^2}{r}\right)\sin\varphi + \frac{\gamma}{2\pi}\log r.$$

The streamline $\quad v_0\left(r - \frac{a^2}{r}\right)\sin\varphi + \frac{\gamma}{2\pi}\log r = C$

The flow picture is



$\gamma < \gamma_c$



2. (a). $f(t) = \dfrac{2}{t^2+4}$

$$f(t) = \frac{\frac{1}{2}}{1+\left(\frac{t}{2}\right)^2} = \frac{1}{2}\,\frac{1}{1+\left(\frac{t}{2}\right)^2}$$

Using the fact that the F.T. of $\dfrac{1}{1+t^2}$ is $\pi e^{-|k|}$

we have

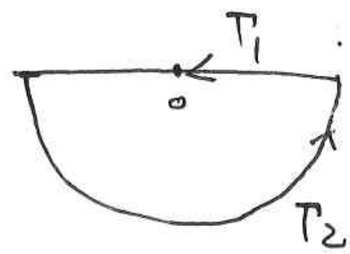
$$\hat{f}(k) = \frac{1}{2}\cdot 2\;\pi\cdot e^{-2|k|} = \pi e^{-2|k|}$$

(b). Since $\widehat{e^{-|x|}} = \dfrac{2}{1+k^2}$

$$\widehat{e^{-2|x|}} = \frac{1}{2}\cdot\frac{2}{1+\left(\frac{k}{2}\right)^2} = \frac{4}{k^2+4}$$

(c) $\hat{f}(k) = \int e^{-ik\eta} \frac{1}{\eta^4+1} d\eta$

For $k>0$, we use



on Γ_1, $\int_R^{-R} e^{-ik\rho} \frac{1}{\rho^4+1} d\rho$

on Γ_2, $\int_{\Gamma_2} \longrightarrow 0$.

$$\eta^4+1=0 \Rightarrow \eta = e^{\frac{(2k+1)\pi}{4} i}, \quad k=0,1,2,3.$$

$$z_3 = e^{\frac{5\pi}{4} i}, \quad z_2 = e^{\frac{7}{4}\pi i}$$

$$\mathrm{Res}\left(e^{-ikz}\frac{1}{z^4+1};\ e^{\frac{5}{4}\pi i}\right) = \frac{e^{-ikz_1}}{4z_1^3}$$

$$\mathrm{Res}\left(e^{-ikz}\frac{1}{z^4+1};\ e^{\frac{7}{4}\pi i}\right) = \frac{e^{-ikz_2}}{4z_2^3}$$

Hence $\int_{-\infty}^{+\infty} e^{-ik\eta}\frac{1}{\eta^4+1} d\eta = -2\pi i \left(\frac{e^{-ike^{\frac{5}{4}\pi i}}}{4z_1^3} + \frac{e^{-ike^{\frac{5}{4}\pi i}}}{4z_2^3} \right)$

$$= \sqrt{2}\pi e^{-\frac{\sqrt{2}}{2}k}\left(\sin\frac{\sqrt{2}}{2}k + \cos\frac{\sqrt{2}}{2}k\right)$$

For $k<0$, we get

$$\hat{f}(k) = \sqrt{2}\pi e^{\frac{\sqrt{2}}{2}k}\left(\sin\frac{\sqrt{2}}{2}k + \cos\frac{\sqrt{2}}{2}k\right).$$

(d). $\hat{f}(k) = \int \frac{1}{(x^2+1)^2} e^{-ikx} \, dx$

$$= \begin{cases} 2\pi i \,\operatorname{Res}\left[\frac{e^{-ikz}}{(z^2+1)^2} ; i\right], & k \le 0 \\ -2\pi i \,\operatorname{Res}\left[\frac{e^{-ikz}}{(z^2+1)^2} ; -i\right], & k > 0. \end{cases}$$

For $k \le 0$, $\operatorname{Res}\left[\frac{e^{-ikz}}{(z^2+1)^2} ; i\right] = \frac{d}{dz}\left(\frac{e^{-ikz}}{(z+i)^2}\right)\Big|_{z=i}$

So $\hat{f}(k) = 2\pi\left(\frac{1-k}{4}\right) e^{k}$

For $k > 0$, $\hat{f}(k) = 2\pi\left(\frac{1+k}{4}\right) e^{-k}$

So $\hat{f}(k) = \frac{\pi}{2}(1+|k|) e^{-|k|}$.

(e). The F.T. of

$e^{-\frac{x^2}{2\sigma^2}}$ is $\sqrt{2\pi}\,\sigma\, e^{-\sigma^2 k^2/2}$

Now $2\sigma^2 = 1 \Rightarrow \sigma = \sqrt{\frac{1}{2}}$,

so $\widehat{e^{-t^2}} = \sqrt{\pi}\, e^{-\frac{k^2}{4}}$

(f). $\widehat{f(t)} = \widehat{\left(-\frac{1}{2}\right)\left(e^{-t^2}\right)'} = -\frac{1}{2}\left((ik)\widehat{e^{-t^2}}\right)$

$$= -\frac{1}{2}(ik)\sqrt{\pi}\, e^{-k^2/4}$$

$$= -\frac{\sqrt{\pi}\, k}{2}\, i\, e^{-k^2/4}$$

3. $\hat{u}(k,t) = \int_{-\infty}^{+\infty} u(x,t)\, e^{-ikx}\, dx$

$$i\hat{u}_t + (ik)^2\hat{u} = 0$$

$$\hat{u}_t + ik^2\hat{u} = 0, \quad \hat{u}(k,0) = \hat{f}(k)$$

$$\hat{u} = \hat{f}(k)\, e^{-ik^2t}$$

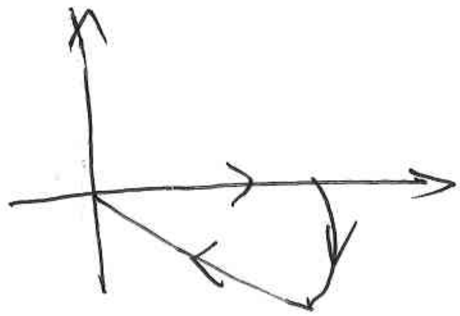
We need to find the I.F.T. of e^{-ik^2t}, which is

$$\frac{1}{2\pi}\int e^{ikx - ik^2t} = \frac{1}{2\pi}\int e^{-it\left(k-\frac{x}{2t}\right)^2} dk\; e^{i\frac{x^2}{4t^2}}$$

$$= \frac{1}{2\pi} e^{i\frac{x^2}{4t^2}} \int e^{-itx^2} dx$$

$$= \frac{1}{2\pi} e^{i\frac{x^2}{4t^2}} \cdot 2\int_0^{+\infty} e^{-ix^2} dx \cdot \frac{1}{\sqrt{t}}$$

The computation of $\int_0^{+\infty} e^{-ix^2} dx$ is by complex contour



$$\Rightarrow \int_0^\infty e^{-ix^2}\,dx \cancel{+} e^{-i\frac{\pi}{4}}\int_\infty^0 e^{-\rho^2}\,d\rho = 0$$

$$\Rightarrow \int_0^\infty e^{-ix^2}\,dx = \frac{\sqrt{\pi}}{2}\,e^{-i\frac{\pi}{4}}$$

So

$$\frac{1}{2\pi}\int e^{ikx-ik^2t}\,dk = \frac{1}{\sqrt{\pi t}}\,e^{-i\pi/4}\,e^{i\frac{x^2}{4t^2}}$$

$$u(x,t) = \frac{1}{\sqrt{\pi t}}\,e^{-i\frac{\pi}{4}}\int f(y)\,e^{i\frac{(x-y)^2}{4t^2}}\,dy$$

4. Recall:

$$\hat{u}(k_1,k_2,t) = \iint e^{-ik_1x-ik_2y}\,u(x,y,t)\,dx\,dy$$

$$u(x,y,t) = \frac{1}{(2\pi)^2}\iint e^{ik_1x+ik_2y}\,\hat{u}(k_1,k_2,t)\,dk_1\,dk_2$$

Take a F.T. $\Rightarrow$

$$\begin{cases}\hat{u}_t = -D(k_1^2+k_2^2)\,\hat{u}\\ \hat{u}(k_1,k_2,0) = \hat{f}\end{cases}$$

So $\hat{u} = \hat{f}(k_1,k_2)\,e^{-D(k_1^2+k_2^2)t}$

So

$$u(x,y,t) = \iint g(x-x', y-y', t)\, f(x', y', t)\, dx'dy'$$

where

$$g = \text{I.F.T. of } e^{-D(k_1^2+k_2^2)t}$$

$$= \frac{1}{(2\pi)^2} \iint e^{ik_1x + ik_2y} e^{-D(k_1^2+k_2^2)t} dk_1 dk_2$$

$$= \frac{1}{2\pi} \int e^{ik_1x - Dk_1^2 t} dk_1 \; \frac{1}{2\pi} \int e^{ik_2y - Dk_2^2 t} dk_2$$

$$= \frac{1}{\sqrt{4\pi D t}} e^{-x^2/(4Dt)} \frac{1}{\sqrt{4\pi D t}} e^{-y^2/(4Dt)}$$

$$= \frac{1}{4\pi D t} e^{-\frac{(x^2+y^2)}{4Dt}}$$

Hence

$$u(x,y,t) = \frac{1}{4\pi D t} \iint e^{-\frac{(x-x')^2+(y-y')^2}{4Dt}} f(x', y')\, dx'dy'$$

5. Look for sol'ns of $e^{ikx + \sigma t}$

$\Rightarrow \quad \sigma = -D_0 k^2 - D_1 k^4$

Hence we get $\quad \hat{u} = \hat{f}(k)\, e^{-(D_0 k^2 + D_1 k^4)t}$

we have

$$u(x,t) = \int g(x-y)\, f(y)\, dy$$

where

$$g = \frac{1}{2\pi} \int e^{ikx - D_1 k^2 t - D_1 k^4 t} \, dk$$

which exists, but is difficult to compute.