A List of Formulas (and Theorems) for MATH301

Part I: Using Complex contour to compute integrals

Theorem:

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}(f(z); z_j)$$

1. Computation of $\mathrm{Res}(f(z); z_0)$, $f(z) = \dfrac{P(z)}{Q(z)}$

- z_0 is a simple pole, $f(z) = \dfrac{P(z)}{Q(z)}$, $Q(z_0) = 0$.

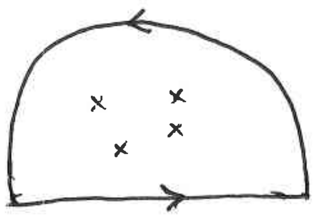
$$\mathrm{Res}(f; z_0) = \frac{P(z_0)}{Q'(z_0)}$$

- z_0 is a pole of order m,

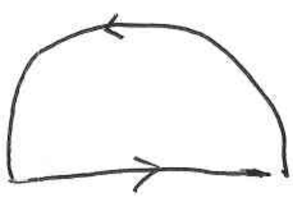
$$\mathrm{Res}(f; z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \Big|_{z=z_0}$$

2. Using contours to compute real integrals

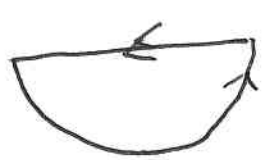
Type 1. $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$, $\quad f(x) = \dfrac{P(x)}{Q(x)}$, $\quad \deg(Q) \geq \deg(P) + 2$



Type 2. $\displaystyle\int_{-\infty}^{+\infty} f(x) e^{i\beta x}\,dx$, $\quad f(x) = \dfrac{P(x)}{Q(x)}$, $\quad \deg Q \geq \deg P + 1$



$\beta > 0$



$\beta < 0$

Type 3. $\int_0^{2\pi} P(\cos\varphi, \sin\varphi)\, d\varphi$, $\int_0^{+\infty} \cos x^m dx$, $\int_0^{\infty} \sin x^m dx$, ...

3. Summation

$$\sum_{k=-\infty}^{+\infty} f(k) = -\sum_{j=1}^{m} \text{Res}\left[f(z)\, \pi \cot(\pi z);\ z_j\right]$$

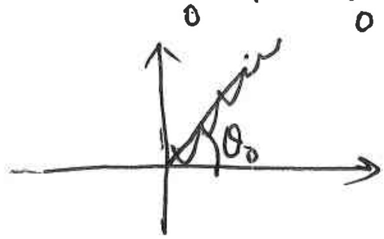
where z_j are poles of $f(z)$.

Part II: Branch Cuts and Multi-Valued Functions

$$z = \rho e^{i\varphi}$$

(1) $\text{Log} z = \log \rho + i\varphi$

Branch cut: $\theta_0 \le \varphi < \theta_0 + 2\pi$, cut at $\varphi = \theta_0$.



(2) $z^\alpha = r^\alpha e^{i\alpha\varphi}$ $\qquad \theta_0 \le \varphi \le \theta_0 + 2\pi$

(3) $\text{Arc}\cos z = -i \log\left(z + (z^2-1)^{\frac{1}{2}}\right)$

(4) $\cosh^{-1} z = \log\left(z + (z^2-1)^{\frac{1}{2}}\right)$

(5) Schwarz-Christoffel Transform:

$$A \int_0^z (z-x_1)^{-\theta_1} (z-x_2)^{-\theta_2}\, dz + B$$

Part III. Conformal Mapping

1) $w(z)=\alpha z+\beta$: translations & rotations

2) Mobius Mapping

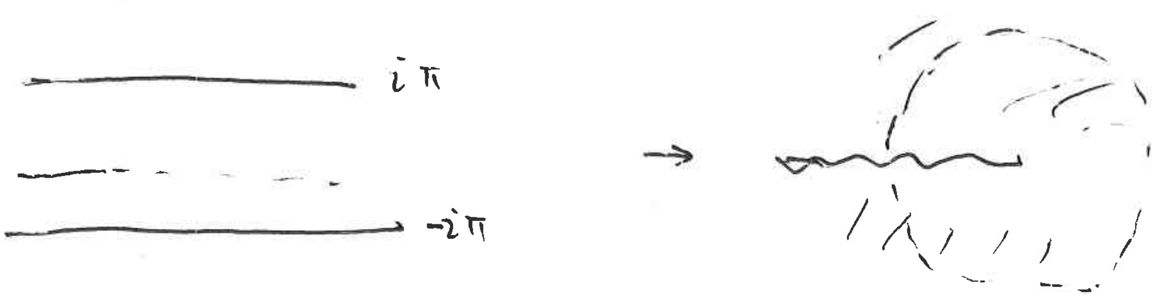
$$w(z)=\frac{az+b}{cz+d}$$

lines, circles > pass through pole $\longleftrightarrow$ lines

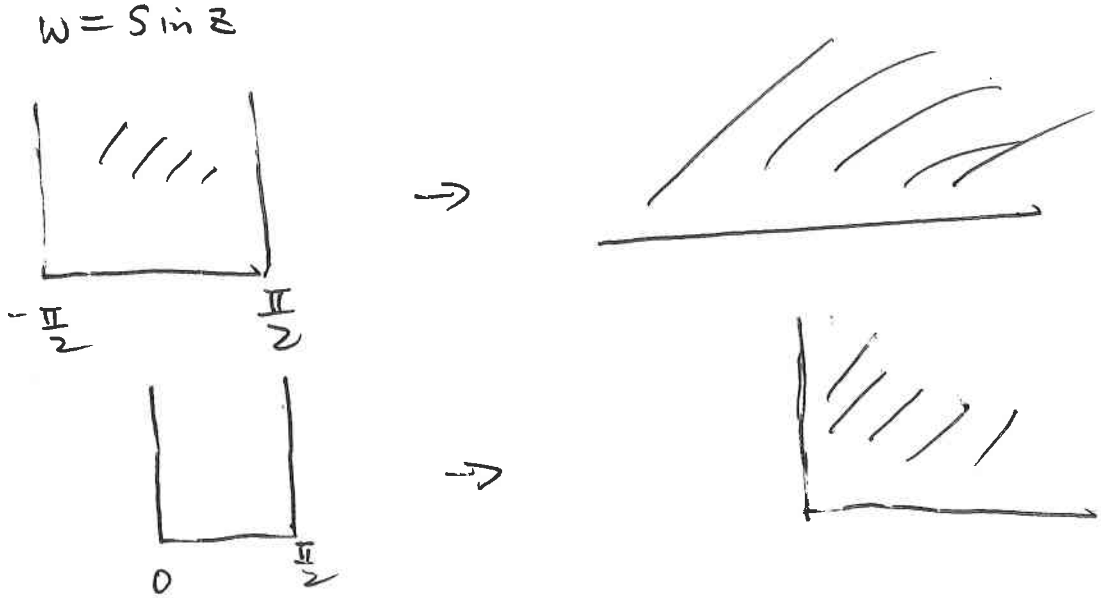
lines, circles > not pass pole $\longleftrightarrow$ circles

symmetric points $\quad (z_2-a)(\bar{z}_1-\bar{a})=R^2.$

3) $w=e^z$

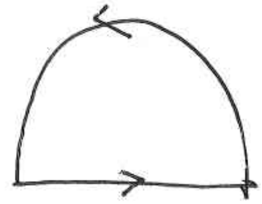


4) $w=\sin z$

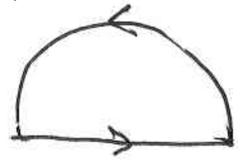


(5) Applications to real integrals.

5.1 $\int_{-\infty}^{+\infty} f(x)\,dx, \quad f(x) = \frac{P(x)}{Q(x)}$



5.2 $\int_{-\infty}^{+\infty} f(x)\, e^{i\beta x}\,dx,$



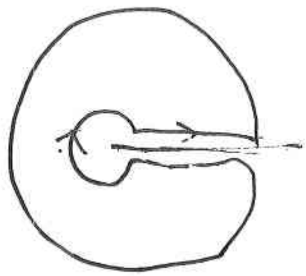
$\beta > 0$

$\beta < 0$

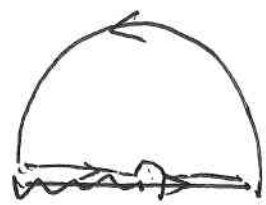
5.3. $\int_{0}^{+\infty} f(x)\,dx$, $f(x)$ even $\Rightarrow$ 5.1

$f(x)$ not even

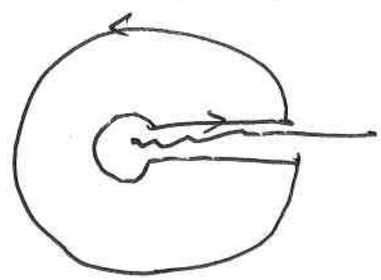
$\int_{C} f(z) \operatorname{Log} z\,dz$



5.4. $\int_{-\infty}^{+\infty} f(x) \log x\,dx$, $f(x)$ even

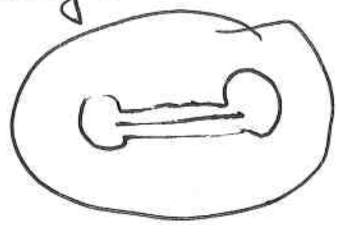


$\int_{0}^{+\infty} f(x) \log x\,dx$, $f(x)$ not even

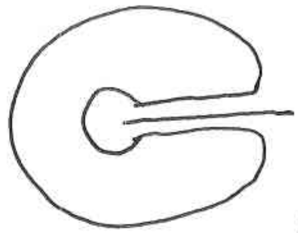


$\int_{C} f(z) \operatorname{Log}^2 z\,dz$

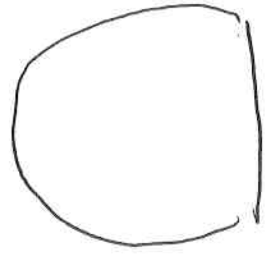
5.5. $\int_{0}^{1} x^{\alpha}(1-x)^{\beta}\,dx$



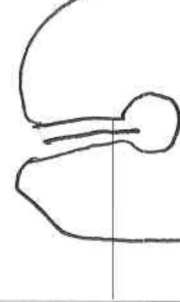
5.6 $\int_{0}^{+\infty} x^{\alpha} f(x)\,dx$



5.7 Inverse Laplace Transform

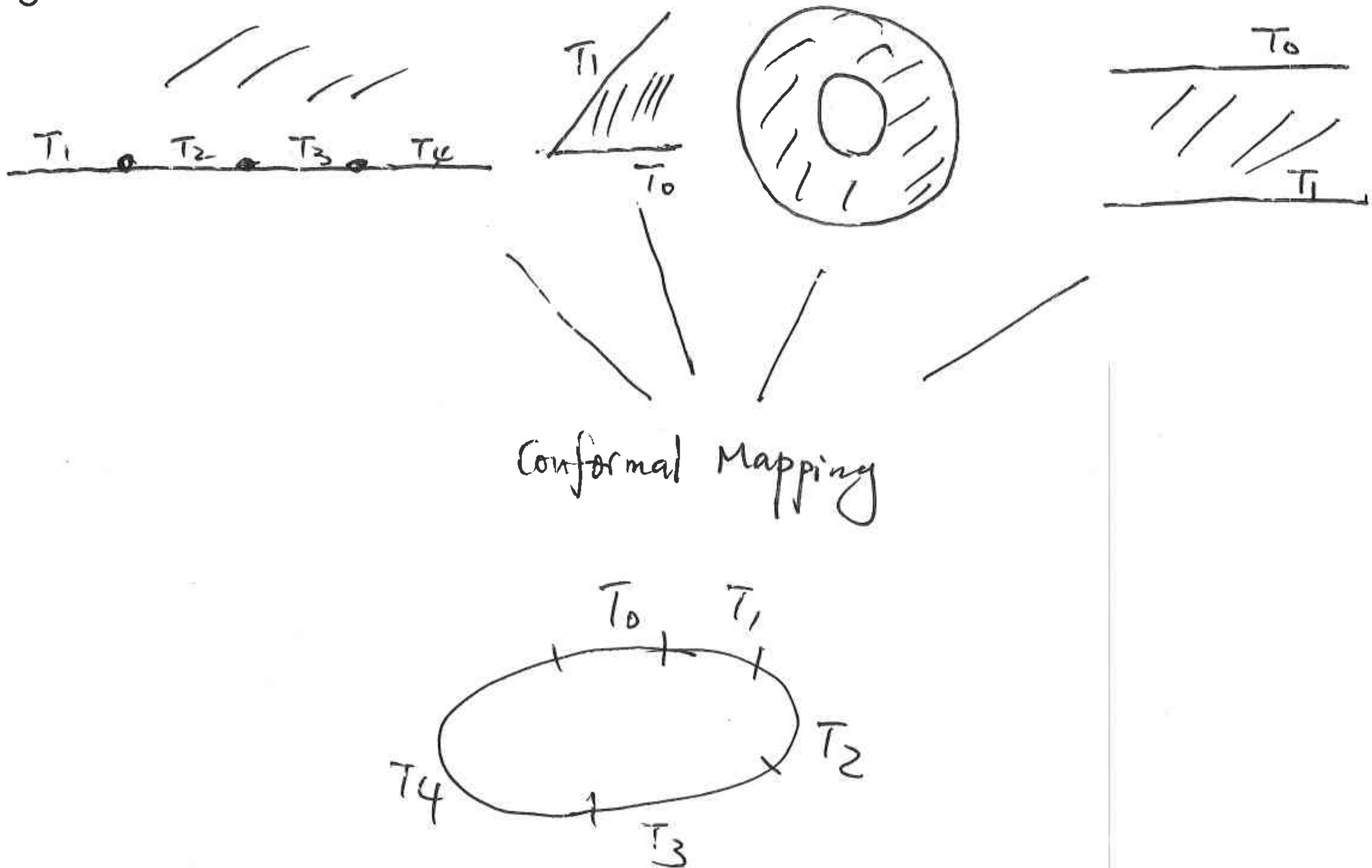


or

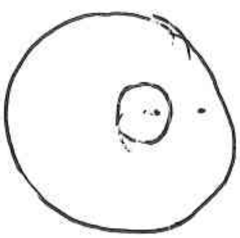


Part IV Conformal Mapping & PDEs

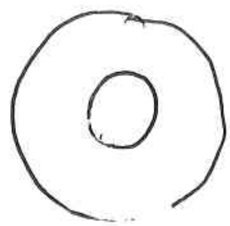
Easy Problems

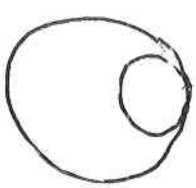


(1)



$$w = \beta \frac{z - z_2}{z - z_1}$$

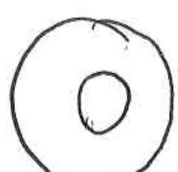


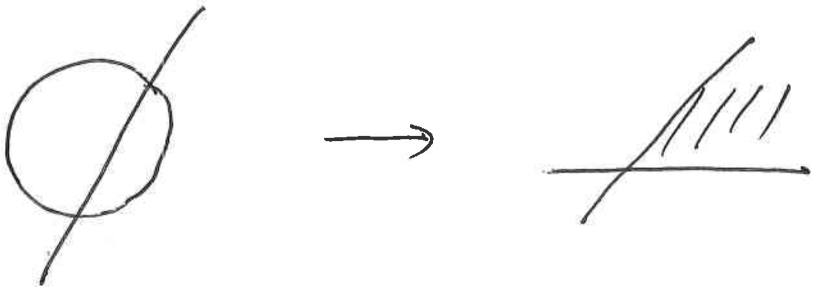


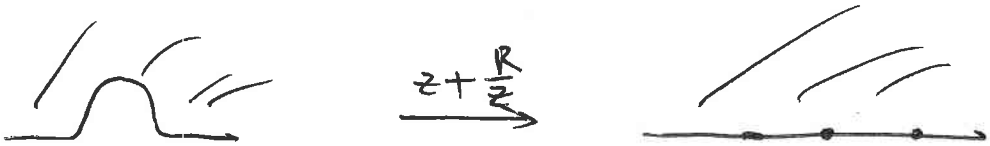
$$w = \beta \frac{z - z_2}{z - z_1}$$

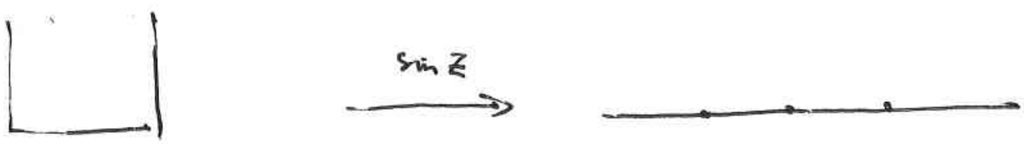
$$w = \beta \frac{z - z_2}{z - z_1}$$

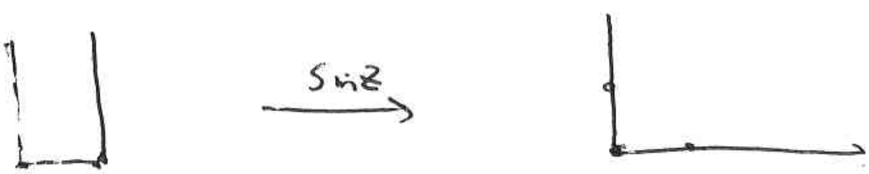
$$w = \beta \frac{z - z_2}{z - z_1}$$





(2) 

(3) 

(4) 

I. Conformal Mapping & Fluid Equation

$$\Omega(z) = \Phi(z) + i\,\Psi(z),$$

$v = \nabla\Phi$, $\quad \Psi$ – streamline function.

$\Psi = C$: streamline

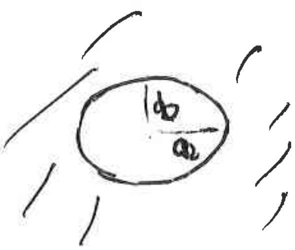
$|\nabla\Phi|^2 = |\Omega'(z)|^2 = 0$: stagnation points

Flow pass a body:

①

$$\Omega(z) = V_0\left(z + \frac{a^2}{z}\right)$$

②



$$\Omega(z) = V_0\left[\zeta + \frac{R^2}{\zeta}\right]$$

$$\zeta + \frac{\tau^2}{\zeta} = z$$

$$R = \frac{a+b}{2}, \quad \tau = \frac{1}{2}(a^2-b^2)^{\frac{1}{2}}$$

Part V: Transforms

Fourier Transforms

$$\hat{f}(k) = \int e^{-ikx} f(x)\, dx$$

$$f(x) = \frac{1}{2\pi}\int e^{ikx}\hat{f}(k)\, dk$$

N-dimensional F.T.

$$\hat{f}(k) = \iiint e^{-ik\cdot x} f(x)\, dx$$

$$f(x) = \frac{1}{(2\pi)^n} \iiint e^{ik\cdot x} \hat{f}(k)\, dk$$

Fourier Transforms Table

$\frac{1}{1+x^2}$	$\pi e^{-\lvert k\rvert}$
$e^{-\lvert x\rvert}$	$\frac{2}{1+k^2}$
$e^{-\frac{x^2}{2\sigma^2}}$	$\sqrt{2\pi}\,\sigma\, e^{-\sigma^2\frac{k^2}{2}}$
$f\left(\frac{x}{b}\right)$	$b\,\hat{F}(bk)$
$\frac{1}{w^2+x^2}$	$\frac{\pi}{w} e^{-w\lvert k\rvert}$
$f * g$	$\hat{f}\,\hat{g}$
f'	$ik\,\hat{f}$

Laplace Trans-form

$$\hat{f}(s) = \int_0^{+\infty} e^{-st} f(t)\, dt$$

$$f(t) = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \hat{f}(s)\, ds$$

Inverse Laplace Transform: $\hat{f}(s) = \dfrac{P(s)}{Q(s)}$, P, Q, polynomial

$$f(t) = \sum \mathrm{Res}(e^{st} \hat{f}(s) ; s_j)$$

$\hat{f}(s) = e^{-cs} \dfrac{P(s)}{Q(s)}$, its inverse Laplace Transform equals.

$$f(t) = u_c(t)\, f(t-c)$$

Laplace Transform Table.

1	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$
$\sin(wt)$	$\dfrac{w}{w^2+s^2}$
$\cos(wt)$	$\dfrac{s}{w^2+s^2}$.
$\int_0^t f(t-s) g(s)\, ds$	$\hat{f}(s)\, \hat{g}(s)$
$u_c(t) f(t-c)$	$e^{-cs} \hat{f}(s)$
f'	$s\hat{f} - f(0)$