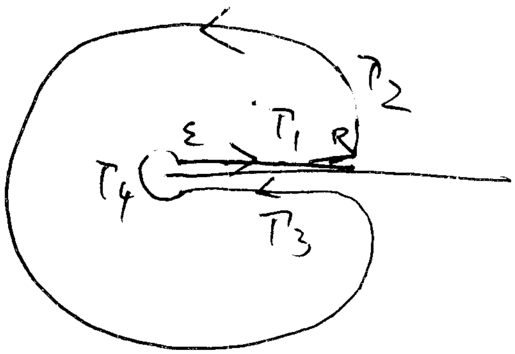
Solutions to Midterm Test

1 Let $f(z) = \dfrac{\mathrm{Log}^2 z}{z^3+1}$ and the contour be



on Γ_1, $z = \rho e^{i0}$, $f(z) = \dfrac{\mathrm{Log}^2 \rho}{\rho^3+1}$.

on Γ_3, $z = \rho e^{i2\pi}$, $f(z) = \dfrac{(\mathrm{Log}\,\rho + i2\pi)^2}{\rho^3+1}$

On Γ_2, $z = Re^{i\varphi}$, $|f(z)| \le c\,\dfrac{\mathrm{Log}\,R}{R^3}$

$$\left|\int_{\Gamma_2} f(z)\,dz\right| \le c\,\frac{\mathrm{Log}\,R}{R^3}\,R \to 0 \text{ as } R \to +\infty$$

on Γ_4, $z = \varepsilon e^{i\varphi}$, $|f(z)| \le c\,|\mathrm{Log}\,\varepsilon|$

$$\left|\int_{\Gamma_4} f(z)\,dz\right| \le c\,\varepsilon|\mathrm{Log}\,\varepsilon| \to 0 \text{ as } \varepsilon \to 0.$$

Residues, $z^3+1=0 \Rightarrow z_1 = e^{i\frac{\pi}{3}}$, $z_2 = -1$, $z_3 = e^{i\frac{5\pi}{3}}$

$$\mathrm{Res}(f, z_j) = \frac{\mathrm{Log}^2 z_j}{3z_j^2} = -\frac{z_j}{3}(\mathrm{Log}\,z_j)^2$$

$$\mathrm{Res}(f, z_1) = \frac{(\mathrm{Log}\,z_1)^2}{3z_1^2} = -\frac{z_1}{3}(\mathrm{Log}\,z_1)^2 = -\frac{e^{i\frac{\pi}{3}}}{3}\left(i\frac{\pi}{3}\right)^2$$

$\mathrm{Res}(f, z_2) = \dfrac{+1}{3}(i\pi)^2$, $\mathrm{Res}(f, z_3) = -\dfrac{e^{i\frac{5\pi}{3}}}{3}\left(i\frac{5\pi}{3}\right)^2$

So

$$\int_0^{+\infty} \frac{\mathrm{Log}^2 \rho}{\rho^3+1}\, d\rho - \int_0^{+\infty} \frac{(\mathrm{Log}\,\rho + i2\pi)^2}{\rho^3+1}\, d\rho$$

$$= 2\pi i \left(\frac{\pi^2}{27} e^{i\frac{\pi}{3}} \; \frac{\pi}{3}^2 + \frac{25}{27}\pi^2 e^{i\frac{5\pi}{3}} \right)$$

Imaginary part $\Rightarrow$

$$-4\pi \int_0^{+\infty} \frac{\mathrm{Log}\,\rho}{\rho^3+1}\, d\rho$$

$$= 2\pi \left(\frac{\pi^2}{27} \cos\frac{\pi}{3} \; \frac{\pi^2}{3} + \frac{25}{27}\pi^2 \cos\frac{\pi}{3} \right)$$

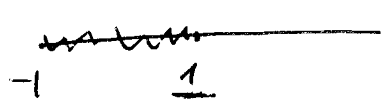
$$\int_0^{+\infty} \frac{\mathrm{Log}\,\rho}{\rho^3+1}\, d\rho = -\frac{2}{27}\pi^2$$

2. $w = (z+1)^{\frac{1}{2}} (z-1)^{-\frac{1}{2}} = r_1^{\frac{1}{2}} r_2^{-\frac{1}{2}} e^{i\frac{1}{2}(\varphi_1 + \varphi_2)}$

$r_1 = |z+1|$, $r_2 = |z-1|$

(a). Let the branch cut be

$0 < \varphi_1 < 2\pi$, $0 < \varphi_2 < 2\pi$



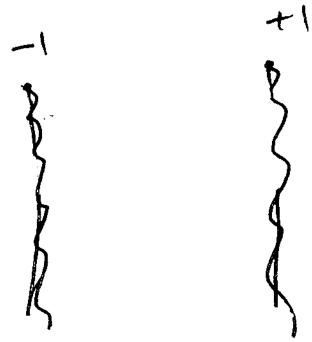
Then f is continuous at $x > 1$ since

$$\left.\begin{array}{ll} \varphi_1 = 0, & \varphi_2 = 0 \\ \varphi_1 = 2\pi, & \varphi_2 = 2\pi \end{array}\right\} \Rightarrow e^{i\frac{1}{2}(0+0)} = e^{i\frac{1}{2}(2\pi+2\pi)}$$

Hence $f(2) = |2+1|^{\frac{1}{2}}|2-1|^{\frac{1}{2}} e^{\frac{i}{2}(0-0)} = \sqrt{3}$

$f(-2) = |-2+1|^{\frac{1}{2}} |-2-1|^{\frac{1}{2}} e^{\frac{i}{2}(\pi+\pi)} = \sqrt{3}$

(b) Define branch cut as:



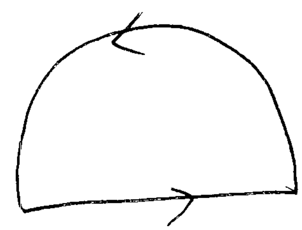
$-\frac{\pi}{2} < \varphi_1 < \frac{3\pi}{2}$

$-\frac{\pi}{2} < \varphi_2 < \frac{3\pi}{2}$

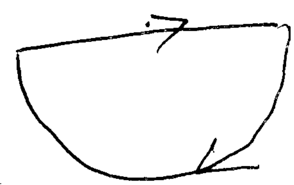
Then $f(2) = |2+1|^{\frac{1}{2}} |2-1|^{\frac{1}{2}} e^{i(0-0)} = \sqrt{3}$

f is continuous at ± 2, and the origin

3. (a). $f(z) = \dfrac{e^{iz}}{z^2+2z+2}$, contour

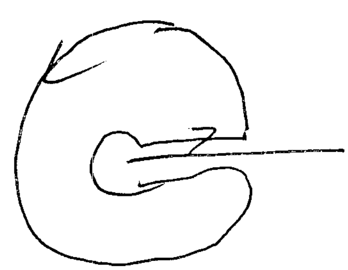


(b) $f(z) = \dfrac{e^{-iz}}{z-i}$, contour

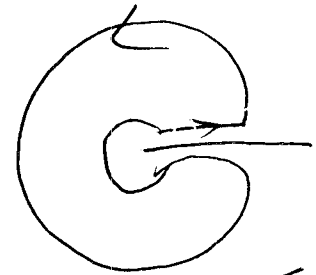


(c) $f(z) = \dfrac{\operatorname{Log} z}{z^2+4}$ contour

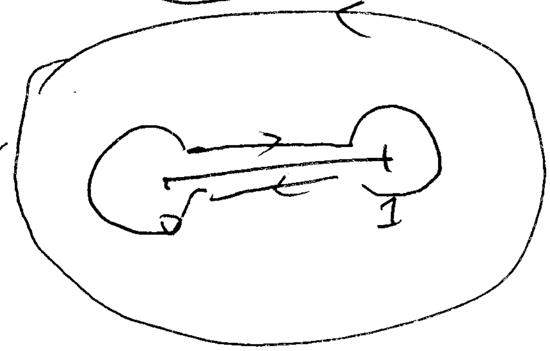
or $f(z) = \dfrac{\operatorname{Log}^2 z}{z^2+4}$, contour



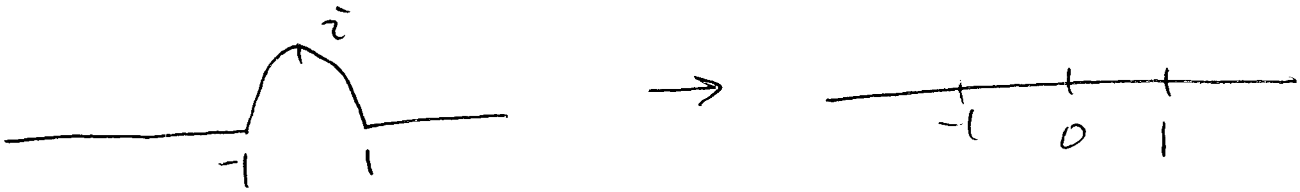
(d) $f(z)=\dfrac{\operatorname{Log} z}{z^2+2z+2}$ Contour



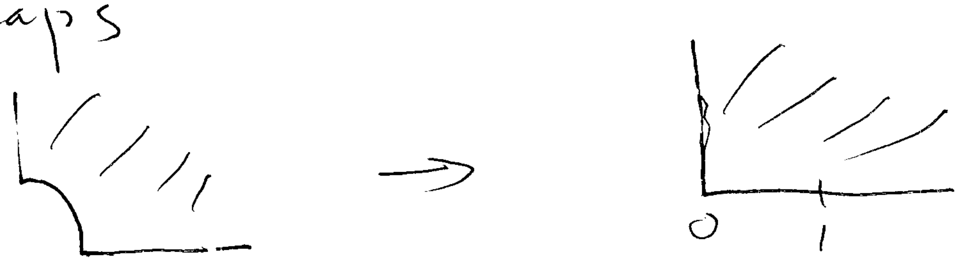
(e) $\displaystyle\int_0^1 \frac{1}{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}}\,dx$, Contour



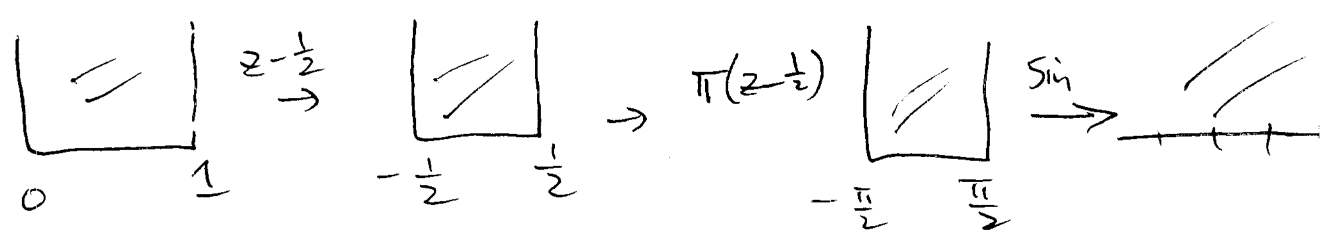
4. (a) The map $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ maps



It also maps



(b).



$w=\sin\pi\left(z-\frac{1}{2}\right)$

(c) we find Mobius transform

z	w
∞	$+1$
i	i
1	-1

$$f(z) = \frac{z+a}{z+b}$$

$$\frac{i+a}{i+b} = i \Rightarrow i+a = -1+ib \Rightarrow a-ib = -1-i$$

$$\frac{1+a}{1+b} = -1 \Rightarrow 1+a = -1-b \Rightarrow a+b = -2$$

$$b = \frac{1+i}{i+1} = 1$$

$$a = -3$$

$$f(z) = \frac{z-3}{z+1}$$