"Conformal Image Warping" by Frederick and Schwartz (see Ref. [9]), offers a somewhat whimsical depiction of the mapping.

Notice that the upper half-circle, where the unknown function ϕ equals 1, is mapped to the positive imaginary axis, whereas the lower half-circle (where $\phi = -1$) corresponds to the negative imaginary axis. Consequently, by the methods of Sec. 3.4 we find the solution in the w-plane to be

$$\psi(u, v) = \frac{2}{\pi} \operatorname{Arg}(w).$$

Hence the solution to the original problem is derived from ψ by the mapping (7):

$$\phi(x, y) = \psi(u(x, y), v(x, y)) = \frac{2}{\pi} \operatorname{Arg}(f(z)) = \frac{2}{\pi} \operatorname{Arg}\left(\frac{1+z}{1-z}\right).$$

A little algebra results in the expression

$$\phi(x, y) = \frac{2}{\pi} \tan^{-1} \frac{2y}{1 - x^2 - y^2},$$

where the value of the arctangent is taken to be between $-\pi/2$ and $\pi/2$. Note that $\phi(x, 0) = 0$, as we would expect from symmetry. ■

With this example as motivation we devote the next few sections to a study of mappings given by analytic functions. The final two sections of the chapter will return us to applications, illustrating the power of this technique in handling many different situations. A table of some of the more useful mappings appears as Appendix II, for the reader's future convenience.

The MATLAB toolbox mentioned in the preface provides an excellent tool for visualizing most of the mappings studied in the chapter.

EXERCISES 7.1

1. Show that the function $w = e^z$ maps the half-strip $x > 0$, $-\pi/2 < y < \pi/2$ onto the portion of the right half w-plane that lies outside the unit circle (see Fig. 7.6). What harmonic function $\psi(w)$ does the w-plane "inherit," via this mapping, from the harmonic function $\phi(z) = x + y$? What harmonic function $\phi(z)$ is inherited from $\psi(w) = u + v$?

2. Suppose that Eqs. (5) and (6) describe a one-to-one analytic mapping. Let $\phi(x, y)$ be a real-valued twice-continuously differentiable function that is carried over in the w-plane to the function

$$\psi(u, v) := \phi(x(u, v), y(u, v)).$$

(a) The *gradient* of $\phi(x, y)$ is the vector $(\partial\phi/\partial x, \partial\phi/\partial y)$; it corresponds to the complex number (recall Sec. 1.3) $\partial\phi/\partial x + i(\partial\phi/\partial y)$. Similarly, the gradient

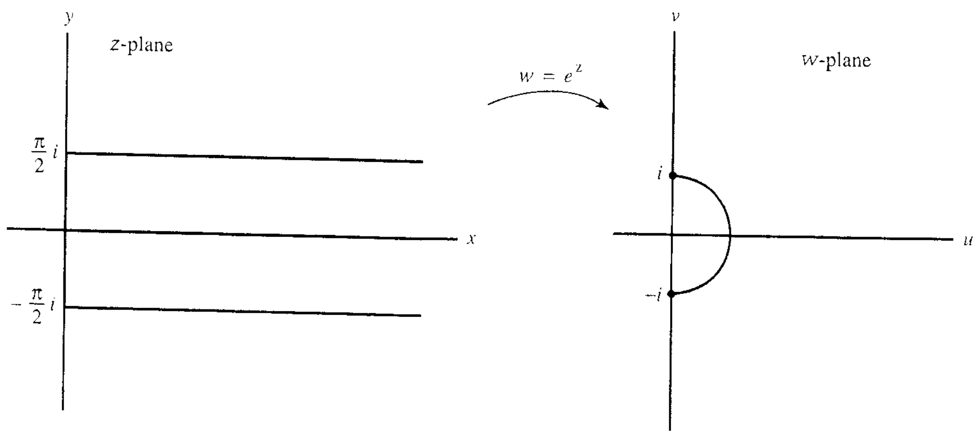

Figure 7.6 Exponential mapping of half-strip.

of ψ corresponds to $\partial\psi/\partial u + i(\partial\psi/\partial v)$. Use the chain rule and the Cauchy-Riemann equations to show that these gradients are related by

$$\frac{\partial\psi}{\partial u} + i\frac{\partial\psi}{\partial v} = \left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right)\overline{\left(\frac{dz}{dw}\right)}.$$

(b) Show that the *Laplacians* of ψ and ϕ are related by

$$\left\{\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2}\right\} = \left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\}\left|\frac{dz}{dw}\right|^2.$$

(c) Show that if $\phi(x, y)$ satisfies Laplace's equation in the z-plane, then ψ satisfies Laplace's equation in the w-plane.

(d) Show that if ϕ satisfies *Helmholtz's* equation,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \Lambda\phi$$

(Λ is a constant), in the z-plane, then ψ satisfies

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} = \Lambda\left|\frac{dz}{dw}\right|^2\psi$$

in the w-plane. (Helmholtz's equation arises in transient thermal analysis.)

3. Find a function ϕ harmonic in the upper half-plane and taking boundary values as indicated in Fig. 7.7. [HINT: Reread Sec. 3.4.]

4. Consider the problem of finding a function ϕ that is harmonic in the right half-plane and takes the values $\phi(0, y) = y/(1 + y^2)$ on the imaginary axis. Observe that the obvious first guess

$$\phi(z) = \text{Im}\,\frac{z}{1 - z^2},$$

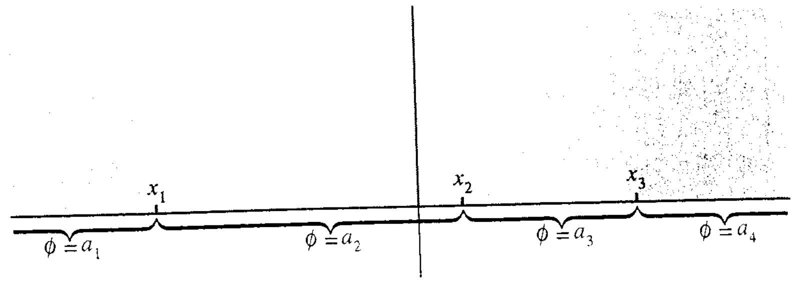

Figure 7.7 Dirichlet problem in Prob. 3.

fails because $z/(1-z^2)$ is *not* analytic at $z = 1$. However, the following strategy can be used.

(a) According to the text, the mappings (7) and (8) provide a correspondence between the right half-plane and the unit disk. (Of course, one should interchange the roles of z and w in the formulas.) Thus the w-plane inherits from $\phi(z)$ a function $\psi(w)$ harmonic in the unit disk. Show that the values of $\psi(w)$ on the unit circle $w = e^{i\theta}$ must be given by

$$\psi(e^{i\theta}) = \frac{\sin\theta}{2}.$$

(b) Argue that the harmonic function $\psi(w)$ must be given by

$$\psi(w) = \frac{1}{2}\,\mathrm{Im}\, w$$

throughout the unit disk.

(c) Use the mappings to carry $\psi(w)$ back to the z-plane, producing the function

$$\phi(z) = \frac{y}{y^2 + (x+1)^2}$$

as a solution of the problem.

5. Use the strategy of Prob. 4 to find a function ϕ harmonic in the right half-plane such that $\phi(0, y) = 1/(y^2 + 1)$.

6. Suppose that the harmonic function $\phi(x, y)$ in the domain D is carried over to the harmonic function $\psi(u, v)$ in the domain D' via the one-to-one analytic mapping $w = f(z)$. Prove that if the normal derivative $\partial\phi/\partial n$ is zero on a curve Γ in D, then the normal derivative $\partial\psi/\partial n$ is zero on the image curve of Γ under f. (The boundary condition $\partial\phi/\partial n = 0$ is known as a *Neumann* condition.) [HINT: $\partial\phi/\partial n$ is the projection of the gradient $(\partial\phi/\partial x) + i(\partial\phi/\partial y)$ onto the normal, and the gradient is orthogonal to the level curves $\phi(x, y) =$ constant.]

7. Suppose that $f(z)$ is analytic and one-to-one. Then, according to the text, you may presume that f^{-1} is also analytic. If x, y, u, v are as in Eqs. (5) and (6), explain the identities

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \qquad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}.$$

7.2 Geometric Considerations

The geometric aspects of analytic mappings split rather naturally into two categories: *local* properties and *global* properties. Local properties need only hold in sufficiently small neighborhoods, while global properties hold throughout a domain. For example, consider the function e^z. It is one-to-one in any disk of diameter less than 2π, and hence it is locally one-to-one, but since $e^{z_1} = e^{z_2}$ when $z_1 - z_2 = 2\pi i$, the function is not globally one-to-one. On the other hand, sometimes local properties can be extended to global properties; in fact, this is the essence of *analytic continuation* (see Sec. 5.8).

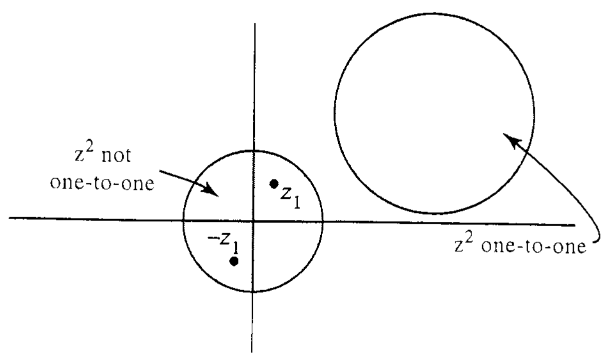

Figure 7.8 Locally one-to-one mapping.

Let us begin our study of local properties by considering "one-to-oneness." As the example e^z shows, a function may be locally one-to-one without being globally one-to-one. (Of course, the opposite situation is impossible.) Furthermore, an analytic function may be locally one-to-one at some points but not at others. Indeed, consider

$$f(z) = z^2.$$

In any open set that contains the origin there will be distinct points z_1 and z_2 such that $z_2 = -z_1$, and hence (since $z_2^2 = z_1^2$) the function f will not be one-to-one. However, around any point other than the origin, we *can* find a neighborhood in which z^2 is one-to-one (any disk that excludes the origin will do; see Fig. 7.8). Thus $f(z) = z^2$ is locally one-to-one at every point other than the origin. An explanation of the exceptional nature of $z = 0$ in this example is provided by the following.

The remainder of this chapter will deal with constructing and applying *specific* conformal mappings.

EXERCISES 7.2

1. For each of the following functions, determine the order m of the zero of the derivative f' at z_0 and show explicitly that the function is not one-to-one in any neighborhood of z_0.

(a) $f(z) = z^2 + 2z + 1$, $z_0 = -1$

(b) $f(z) = \cos z$, $z_0 = 0, \pm\pi, \pm 2\pi, \ldots$

(c) $f(z) = e^{z^3}$, $z_0 = 0$

2. Prove that if $w = f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$, then $z = f^{-1}(w)$ is analytic at $w_0 = f(z_0)$, and

$$\frac{df^{-1}}{dw}(w) = \frac{1}{\dfrac{df}{dz}(z)}$$

for $w = w_0$, $z = z_0$. [HINT: Theorem 1 guarantees that $f^{-1}(w)$ exists near w_0 and Theorem 3 implies that $f^{-1}(w)$ is continuous. Now generalize the proof in Sec. 3.2.]

3. What happens to angles at the origin under the mapping $f(z) = z^\alpha$ for $\alpha > 1$? For $0 < \alpha < 1$?

4. Use the open mapping theorem to prove the maximum-modulus principle.

5. Find all functions $f(z)$ analytic in $D : |z| < 1$ that assume only pure imaginary values in D.

6. If f is analytic at z_0 and $f'(z_0) \neq 0$, show that the function $g(z) = \overline{f(z)}$ preserves the magnitude, but reverses the orientation, of angles at z_0.

7. Show that the mapping $w = z + 1/z$ maps circles $|z| = \rho$ $(\rho \neq 1)$ onto ellipses

$$\frac{u^2}{\left(\rho + \dfrac{1}{\rho}\right)^2} + \frac{v^2}{\left(\rho - \dfrac{1}{\rho}\right)^2} = 1.$$

8. Let f be analytic at z_0 with $f'(z_0) \neq 0$. By considering the difference quotient, argue that "infinitesimal" lengths of segments drawn from z_0 are magnified by the factor $|f'(z_0)|$ under the mapping $w = f(z)$.

9. Let $w = f(z)$ be a one-to-one analytic mapping of the domain D onto the domain D', and let $A' = \text{area}\,(D')$. Using Prob. 8, argue the plausibility of the formula

$$A' = \iint_D |f'(z)|^2\, dx\, dy.$$

10. Why is it impossible for D to be the whole plane in the Riemann mapping theorem? [HINT: Appeal to Liouville's theorem.]

11. Describe the image of each of the following domains under the mapping $w = e^z$.

(a) the strip $0 < \operatorname{Im} z < \pi$

(b) the slanted strip between the two lines $y = x$ and $y = x + 2\pi$

(c) the half-strip $\operatorname{Re} z < 0$, $0 < \operatorname{Im} z < \pi$

(d) the half-strip $\operatorname{Re} z > 0$, $0 < \operatorname{Im} z < \pi$

(e) the rectangle $1 < \operatorname{Re} z < 2$, $0 < \operatorname{Im} z < \pi$

(f) the half-planes $\operatorname{Re} z > 0$ and $\operatorname{Re} z < 0$

12. Let $P(z) = (z - \alpha)(z - \beta)$, and let L be any straight line through $(\alpha + \beta)/2$. Prove that P is one-to-one on each of the open half-planes determined by L.

13. Describe the image of each of the following domains under the mapping $w = \cos z = \cos x \cosh y - i \sin x \sinh y$. [HINT: Consider the image of the boundary in each case.]

(a) the half-strip $0 < \operatorname{Re} z < \pi$, $\operatorname{Im} z < 0$

(b) the half-strip $0 < \operatorname{Re} z < \dfrac{\pi}{2}$, $\operatorname{Im} z > 0$

(c) the strip $0 < \operatorname{Re} z < \pi$

(d) the rectangle $0 < \operatorname{Re} z < \pi$, $-1 < \operatorname{Im} z < 1$

14. Prove that if f has a simple pole at z_0, then there exists a punctured neighborhood of z_0 on which f is one-to- one.

15. A domain D is said to be *convex* if for any two points z_1, z_2 in D, the line segment joining z_1 and z_2 lies entirely in D. Prove the *Noshiro-Warschawski theorem*: Let f be analytic in a convex domain D. If $\operatorname{Re} f'(z) > 0$ for all z in D, then f is one-to-one in D. [HINT: Write $f(z_2) - f(z_1)$ as an integral of f'.]

16. (For students who have read Sec. 4.4a) Argue that a one-to-one analytic function will map simply connected domains to simply connected domains.

7.3 Möbius Transformations

The problem of finding a one-to-one analytic function that maps one domain onto another can be quite perplexing, so it is worthwhile to investigate a few elementary mappings in order to compile some rules of thumb that we can draw upon. The basic properties of *Möbius transformations*,† which we shall investigate in this section, constitute an essential portion of every analyst's bag of tricks. (Some of these mappings were previewed in Exercises 2.1.)

†In 1865 August Möbius (1790-1860) described the *Möbius strip*, a piece of paper that has only one side and one edge.

From the geometric properties of Möbius transformations that we have learned, we can conclude that (11) maps $|z| = 1$ onto *some* straight line through the origin. To see *which* straight line, we plug in $z = i$ and find that the point

$$w = \frac{i+1}{i-1} = -i$$

also lies on the line. Hence the image of the circle under f_1 must be the imaginary axis.

To see which half-plane is the image of the interior of the circle, we check the point $z = 0$. It is mapped by (11) to the point $w = -1$ in the *left* half-plane. This is not what we want, but it can be corrected by a final rotation of π, yielding

$$w = f(z) = -\frac{z+1}{z-1} = \frac{1+z}{1-z} \tag{12}$$

as an answer to the problem. (Of course, any subsequent vertical translation or magnification can be permitted.) Observe that (12) is precisely the mapping that was introduced in Example 1, Sec. 7.1, to solve a thermal problem, and we have thus verified the claims made there. ■

EXERCISES 7.3

1. Find a linear transformation mapping the circle $|z| = 1$ onto the circle $|w - 5| = 3$ and taking the point $z = i$ to $w = 2$.

2. What is the image of the strip $0 < \operatorname{Im} z < 1$ under the mapping $w = (z - i)/z$?

3. Discuss the image of the circle $|z - 2| = 1$ and its interior under the following transformations.

(a) $w = z - 2i$ **(b)** $w = 3iz$ **(c)** $w = \dfrac{z-2}{z-1}$

(d) $w = \dfrac{z-4}{z-3}$ **(e)** $w = \dfrac{1}{z}$

4. Find a Möbius transformation mapping the lower half-plane to the disk $|w + 1| < 1$. [HINT: Do it in steps.]

5. Find a Möbius transformation mapping the unit disk $|z| < 1$ onto the right half-plane and taking $z = -i$ to the origin.

6. A *fixed point* of a function $f(z)$ is a point z_0 satisfying $f(z_0) = z_0$. Show that a Möbius transformation $f(z)$ can have at most two fixed points in the complex plane unless $f(z) \equiv z$.

7. Find the Möbius transformation that maps $0, 1, \infty$ to the following respective points.

(a) $0, i, \infty$ **(b)** $0, 1, 2$ **(c)** $-i, \infty, 1$ **(d)** $-1, \infty, 1$

8. What is the image, under the mapping $w = (z + i)/(z - i)$, of the third quadrant?

9. What is the image of the sector $-\pi/4 < \operatorname{Arg} z < \pi/4$ under the mapping $w = z/(z-1)$?

10. Find a conformal map of the semidisk $|z| < 1$, $\operatorname{Im} z > 0$, onto the upper half-plane. [HINT: Combine a Möbius transformation with the mapping $w = z^2$. Make sure you cover the entire upper half-plane.]

11. Map the shaded region in Fig. 7.21 conformally onto the upper half-plane. [HINT: Use a Möbius transformation to map the point 2 to ∞. Argue that the image region will be a *strip*. Then use the exponential map.]

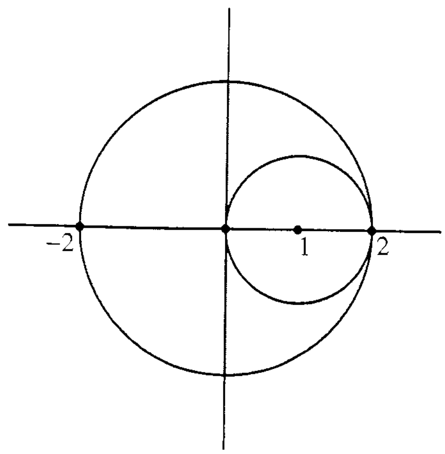

Figure 7.21 Region for Prob. 11.

12. Find a Möbius transformation that takes the half-plane depicted in Fig. 7.22 onto the unit disk $|w| < 1$.

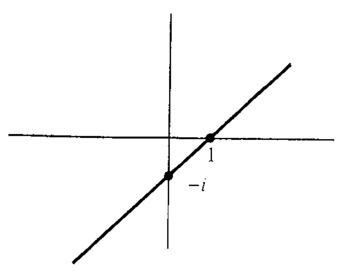

Figure 7.22 Region for Prob. 12.

13. (*Smith Chart*) The *impedance* Z of an electrical circuit oscillating at a frequency ω is a complex number, denoted $Z = R + iB$, which characterizes the voltage-current relationship of the circuit; recall Sec. 3.6. In practice R can take any value from 0 to ∞ and B can take any value from $-\infty$ to ∞. Thus the usual representation of

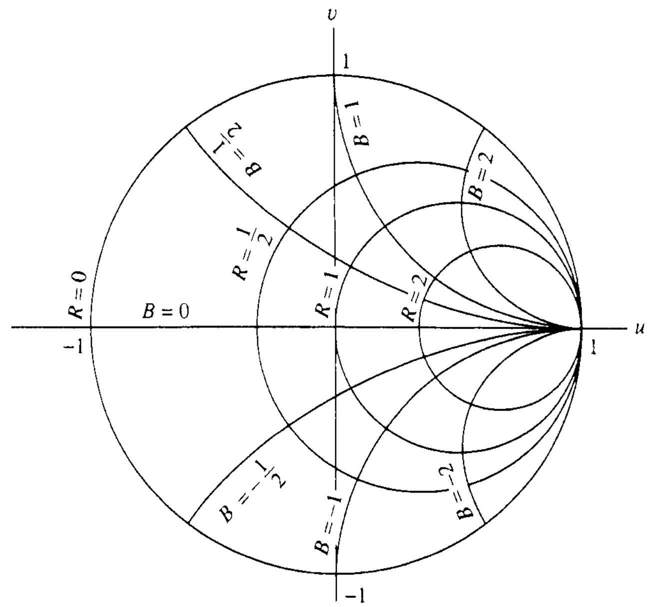

Figure 7.23 Smith chart.

Z as a point in the complex plane becomes unwieldy (inasmuch as the entire right half-plane comes into play). The *Smith chart* provides a more compact graphical description, displaying the entire range of impedances within the unit circle. The impedance Z is depicted as the point

$$W = \frac{Z-1}{Z+1}.$$

This mapping (its inverse, actually) is portrayed in Figs. 7.4 and 7.5. W is also known as the *reflection coefficient* corresponding to Z.

(a) Show that the circles in the Smith chart depicting the lines $\operatorname{Re} Z = R =$ constant, indicating constant-resistance contours, have the equations

$$\left(u - \frac{R}{1+R}\right)^2 + v^2 = \frac{1}{(1+R)^2}.$$

(b) Show that the circles in the Smith chart depicting the lines $\operatorname{Im} Z = B =$ constant, indicating constant-reactance contours, have the equations

$$(u-1)^2 + \left(v - \frac{1}{B}\right)^2 = \frac{1}{B^2}.$$

(See Fig. 7.23.)

14. If a circuit with impedance Z is connected to a length ℓ of *transmission line* with "phase constant" β and a "characteristic impedance" of unity, then the new configuration has a transformed impedance Z' given by

$$Z' = \frac{Z\cos\beta\ell + i\sin\beta\ell}{\cos\beta\ell + iZ\sin\beta\ell}.$$

Show that the Smith chart point depicting Z' can be obtained from the Smith chart point depicting Z by a clockwise rotation of $2\beta\ell$ radians about the origin.†

15. Show that the transformation (5) maps lines not passing through the origin onto circles passing through the origin. [HINT: The equation of such a line is $Ax + By = C$, with $C \neq 0$. Solve

$$z = x + iy = 1/w = 1/(u + iv) \tag{13}$$

for x and y in terms of u and v and substitute. Show that the result can be expressed in the form

$$u^2 + v^2 - \frac{A}{C}u + \frac{B}{C}v = 0.] \tag{14}$$

16. Show that the transformation (5) maps circles passing through the origin onto lines not passing through the origin. [HINT: Use the preceding problem.]

17. Show that the transformation (5) maps circles not passing through the origin onto circles not passing through the origin. [HINT: The equation of such circles is

$$x^2 + y^2 + Ax + By = C, \qquad \text{with} \quad C \neq 0.$$

Substitute the expressions for x and y derived from (13) to obtain

$$u^2 + v^2 - \frac{A}{C}u + \frac{B}{C}v = \frac{1}{C}.]$$

7.4 Möbius Transformations, Continued

We shall now explore some additional properties of Möbius transformations that enhance their usefulness as conformal mappings. These are the group properties, the cross-ratio formula, and the symmetry property.

Given any Möbius transformation

$$w = f(z) = \frac{az + b}{cz + d} \qquad (ad \neq bc), \tag{1}$$

its inverse $f^{-1}(w)$ can be found by simply solving Eq. (1) for z in terms of w. This computation yields

$$z = f^{-1}(w) = \frac{dw - b}{-cw + a},$$

†P. H. Smith patented the Smith chart in the late 1930s. It is the only known conformal mapping to be protected by copyright!