(c) Show that if the Möbius transformations T_1 and T_2 are associated as in part (b) with the elements

$$S_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$$

of $\mathcal{L}$, then the composition $T_1 \circ T_2$ is associated with the product matrix $S_1 S_2$.

23. Let z be fixed with $\operatorname{Re} z \geq 0$, and let

$$T_0(w) = \frac{a_0}{z + a_0 + b_1 + w}, \quad T_k(w) = \frac{a_k}{z + b_{k+1} + w} \quad (k = 1, 2, \ldots, n-1)$$

be a sequence of Möbius transformations such that each a_k is real and positive and each b_k is pure imaginary or zero. Prove, by induction, that the composition

$$\zeta = S(w) := T_0 \circ T_1 \circ \cdots \circ T_{n-2} \circ T_{n-1}(w)$$

maps the half-plane $\operatorname{Re} w > 0$ onto a region contained in the disk $|\zeta - \frac{1}{2}| < \frac{1}{2}$.

24. Let $P(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$ be a polynomial of degree $n > 0$ with complex coefficients $c_k = p_k + iq_k$, $k = 1, 2, \ldots, n$. Set $Q(z) := p_1 z^{n-1} + iq_2 z^{n-2} + p_3 z^{n-3} + iq_4 z^{n-4} + \cdots$. Prove *Wall's criterion* that if $Q(z)/P(z)$ can be written in the form

$$\frac{Q(z)}{P(z)} = \cfrac{a_0}{z + a_0 + b_1 + \cfrac{a_1}{z + b_2 + \cfrac{a_2}{z + b_3 + \ddots + \cfrac{a_{n-1}}{z + b_n}}}},$$

where each a_k is real and positive and each b_k is pure imaginary or zero, then all the zeros of $P(z)$ have negative real parts. [HINT: Write $Q(z)/P(z) = T_0 \circ T_1 \circ \cdots \circ T_{n-1}(0)$, where the transformations T_k are defined as in Prob. 23.]

25. Prove that $P(z) = z^3 + 3z^2 + 6z + 6$ has all its zeros in the left half-plane by applying the result of Prob. 24. [HINT: Use ordinary long division to obtain the representation for $Q(z)/P(z)$.]

7.5 The Schwarz-Christoffel Transformation

We have seen that a function $f(z)$ is conformal at every point at which it is analytic and its derivative is nonzero. It is instructive to analyze what happens at certain isolated points where these conditions are not met. For concreteness, let x_1 be a fixed point on the real axis and let $f(z)$ be a function whose *derivative* $f'(z)$ is given by $(z - x_1)^\alpha$ for some real α satisfying $-1 < \alpha < 1$. [To be precise, we shall take the argument of $z - x_1$ to lie between $-\pi/2$ and $3\pi/2$, introducing a branch cut vertically downward from x_1; see Fig. 7.31(a).] We are going to use the equation

$$f'(z) = (z - x_1)^\alpha \tag{1}$$

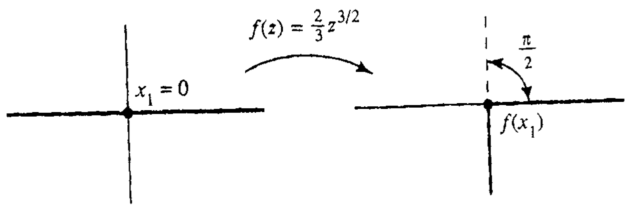

Figure 7.33 Mapping of x-axis by $f(z) = \frac{2}{3}z^{3/2}$.

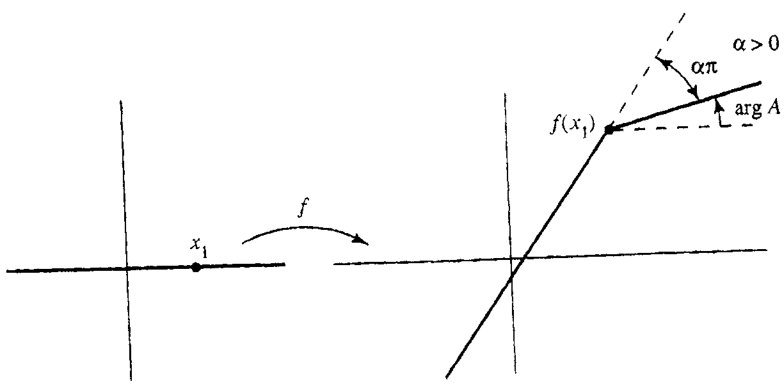

Figure 7.34 Mapping for Eq. (2).

for some complex constant A ($\neq 0$), then

$$\arg f'(z) = \arg A + \alpha \arg (z - x_1)\,,$$

and the mapping can be visualized by rotating Fig. 7.32(b) by an amount arg A; see Fig. 7.34. In particular, the angle made by the image of the interval (x_1, ∞) is now arg A, but the angle of the turn at $f(x_1)$ is unchanged.

The next generalization is to consider a mapping given by a function f with a derivative of the form

$$f'(z) = A\,(z - x_1)^{\alpha_1}\,(z - x_2)^{\alpha_2} \cdots (z - x_n)^{\alpha_n}\,; \tag{3}$$

here A ($\neq 0$) is a complex constant, each α_i lies between -1 and $+1$, and the (real) x_i satisfy

$$x_1 < x_2 < \cdots < x_n\,.$$

(As before we take the argument of each $z - x_i$ to be between $-\pi/2$ and $3\pi/2$.) What does this mapping f do to the real axis?

From the equation

$$\arg f'(z) = \arg A + \alpha_1 \arg (z - x_1) + \alpha_2 \arg (z - x_2) + \cdots + \alpha_n \arg (z - x_n)$$

and the previous discussion we see that the images of the intervals $(-\infty, x_1)$, (x_1, x_2), $\ldots$, (x_n, ∞) are each portions of straight lines, making angles measured counterclockwise from the horizontal in accordance with the following prescription:

Interval	*Angle of image*
$(-\infty, x_1)$	$\arg A + \alpha_1\pi + \alpha_2\pi + \cdots + \alpha_n\pi$
(x_1, x_2)	$\arg A + \alpha_2\pi + \cdots + \alpha_n\pi$
$\vdots$	$\vdots$
(x_{n-1}, x_n)	$\arg A + \alpha_n\pi$
(x_n, ∞)	$\arg A.$

Hence as z traverses the real axis from left to right $f(z)$ generates a polygonal path whose tangent at the point $f(x_i)$ *makes a right turn through the angle* $\alpha_i\pi$; see Fig. 7.35.

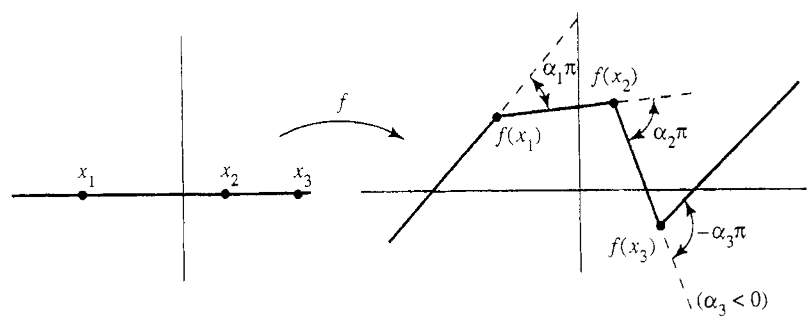

Figure 7.35 Mapping for Eq. (3).

Now if the function $f(z)$ satisfies Eq. (3) it is, a priori, differentiable and hence analytic on the complex plane with the exception of the (downward) branch cuts from the points x_i. So for any z in the upper half-plane we can set

$$g(z) := \int_\Gamma f'(\zeta)\, d\zeta, \tag{4}$$

where Γ is, for definiteness, the straight line segment from 0 to z, and conclude then that $f(z) = g(z) + B$ for some constant B. In particular, we can write

$$f(z) = A \int_0^z (\zeta - x_1)^{\alpha_1} (\zeta - x_2)^{\alpha_2} \cdots (\zeta - x_n)^{\alpha_n}\, d\zeta + B. \tag{5}$$

Functions of the form (5) are known as *Schwarz-Christoffel transformations.*[†] We have seen that such transformations map the real axis onto a polygonal path. Now one of the most important problems in conformal mapping applications is the construction

[†]Hermann Amandus Schwarz (1842-1921), Elwin Bruno Christoffel (1829-1900).

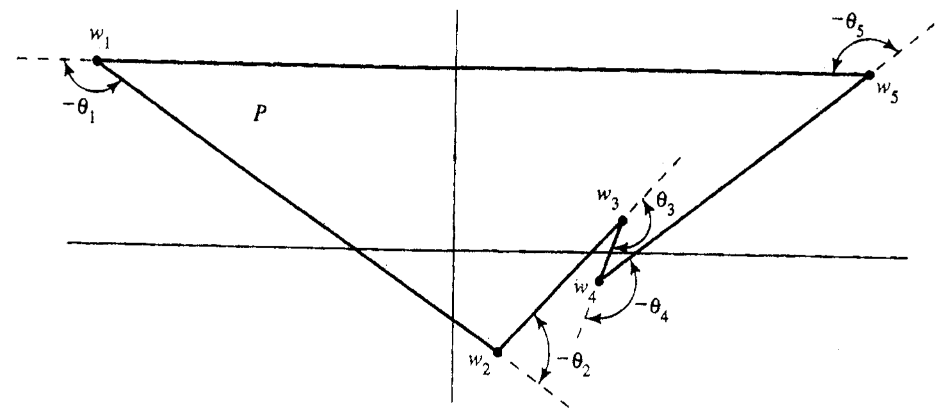

Figure 7.36 Positively oriented polygon (θ_2, θ_3, θ_4, θ_5 are negative).

of a one-to-one analytic function carrying the upper half-plane to the interior of a *given* polygon. We thus turn to the task of tailoring a Schwarz-Christoffel transformation to accomplish this.

To be specific, let the polygon P have vertices at the consecutive points w_1, w_2, $\ldots$, w_n taken in *counterclockwise* order, giving P a positive orientation, as in Fig. 7.36. In traversing the polygon we make a right turn at vertex w_i through the angle θ_i. Thus each angle lies between $-\pi$ and π and a negative value of θ_i indicates a left turn. The *net* rotation for a counterclockwise tour must be 2π radians to the left:

$$\theta_1 + \theta_2 + \cdots + \theta_n = -2\pi. \tag{6}$$

To map the x-axis onto P with a Schwarz-Christoffel transformation $w = g(z)$ we begin by picking real points $x_1, x_2, \ldots, x_{n-1}$ as the preimages of the vertices $w_1, w_2, \ldots, w_{n-1}$, and presume that both $x = -\infty$ and $x = \infty$ are the preimages of w_n; see Fig. 7.37. From the discussion of Eq. (5) it follows that the function

$$g(z) := \int_0^z (\zeta - x_1)^{\theta_1/\pi} (\zeta - x_2)^{\theta_2/\pi} \cdots (\zeta - x_{n-1})^{\theta_{n-1}/\pi} \, d\zeta \tag{7}$$

maps the real axis onto *some* polygon P'. Although P' may not be the desired polygon P, it does have the proper right-turn angles $\alpha_i \pi = \theta_i$ at the corners $g(x_i)$ for $i = 1, 2, \ldots, n-1$; and since the initial and final segments intersect at $g(\pm\infty)$, the right turn at this final vertex must match the angle θ_n (because both are given by $-2\pi - \theta_1 - \theta_2 - \cdots - \theta_{n-1}$).

Now because P' has the same angles as P, by adjusting the lengths of the sides of P' we can make it *geometrically similar* to P. And it seems quite plausible that we could accomplish this by adjusting the points $x_1, x_2, \ldots, x_{n-1}$; after all, they determine where the corners of P' lie. Then, with the use of a rotation, a magnification, and a translation—in other words, a linear transformation—we could make these similar polygons coincide.

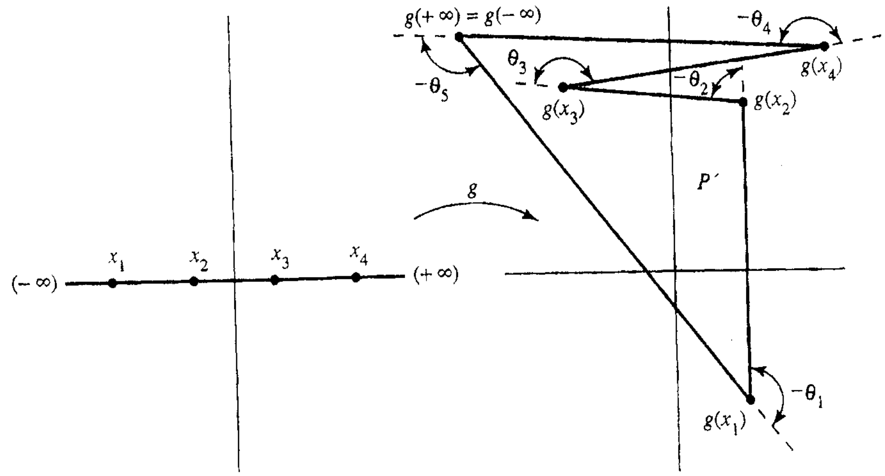

Figure 7.37 Mapping for Eq. (7).

Summarizing, we are led to speculate that with an appropriate choice of the constants we can construct a function

$$\begin{aligned} f(z) &= Ag(z) + B \\ &= A \int_0^z (\zeta - x_1)^{\theta_1/\pi} (\zeta - x_2)^{\theta_2/\pi} \cdots (\zeta - x_{n-1})^{\theta_{n-1}/\pi} \, d\zeta + B, \end{aligned} \tag{8}$$

that is, a Schwarz-Christoffel transformation, which maps the real axis onto the perimeter of a given polygon P, with the correspondences

$$f(x_1) = w_1, \quad f(x_2) = w_2, \quad \ldots, \quad f(x_{n-1}) = w_{n-1}, \quad f(\infty) = w_n. \tag{9}$$

Moreover, if our speculations are valid, we can use conformality and connectivity arguments to show that f maps the upper half-plane to the interior of P, as was requested; for observe that if γ is a segment as indicated in Fig. 7.38, conformality requires that its image, γ', have a tangent that initially points inward as shown, and connectivity completes the argument (assuming one-to-oneness). The whole story about Schwarz-Christoffel transformations is given in Theorem 7, whose proof can be found in the references.

Theorem 7. Let P be a positively oriented polygon having consecutive corners at $w_1, w_2, \ldots, w_n$ with corresponding right-turn angles θ_i $(i = 1, 2, \ldots, n)$. Then there exists a function of the form (8) that is a one-to-one conformal map from the upper half-plane onto the interior of P. Furthermore, the correspondences (9) hold.

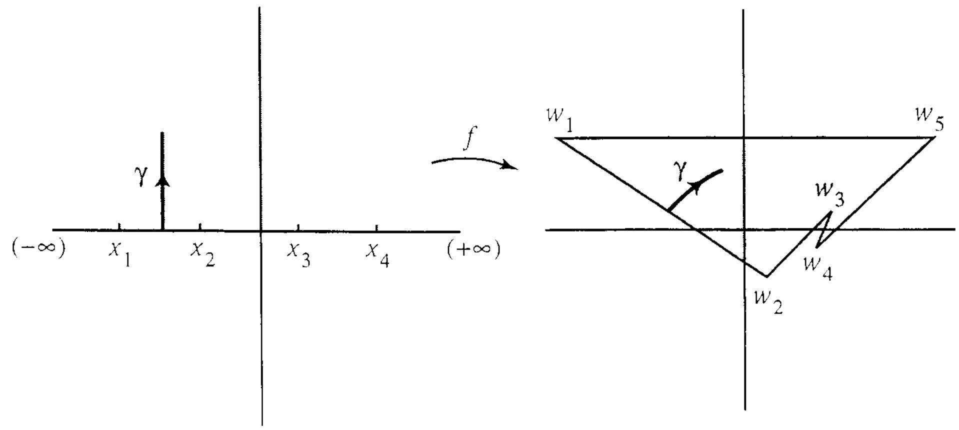

Figure 7.38 The upper half-plane is mapped to the interior of P.

Before we illustrate the technique, we must make two remarks. First, recall that in constructing the map we have three "degrees of freedom" at our disposal (from the Riemann mapping theorem). Thus we can specify three points on the real axis to be the preimages of three of the w_i. However, formula (9) already designates ∞ as the preimage of w_n, so we are free to choose only, say, x_1 and x_2, and the other x_i are then determined.

Second, to get a closed-form expression for the mapping we must be able to compute the integral in Eq. (8). A glance through a standard table of integrals shows that this is hopeless for $n > 4$ and not always possible even for smaller n. Numerical integration, however, is always feasible. In Appendix I, L. N. Trefethen and T. Driscoll discuss how to implement these computations, and provide reference to their readily accessible software package.

Example 1

Derive a Schwarz-Christoffel transformation mapping the upper half-plane onto the triangle in Fig. 7.39.

Solution. The right turns are through angles $\theta_1 = \theta_2 = -3\pi/4$, $\theta_3 = -\pi/2$. Hence, choosing $x_1 = -1$ and $x_2 = 1$ we have

$$\begin{aligned} f(z) &= A\int_0^z (\zeta+1)^{-3/4}(\zeta-1)^{-3/4}\,d\zeta + B \\ &= A\int_0^z \left(\zeta^2-1\right)^{-3/4}\,d\zeta + B. \end{aligned}$$

The integration must be performed numerically. To evaluate the constants we compute

$$f(x_1) = f(-1) = A\int_0^{-1}\left(\zeta^2-1\right)^{-3/4}\,d\zeta + B = A\eta + B,$$

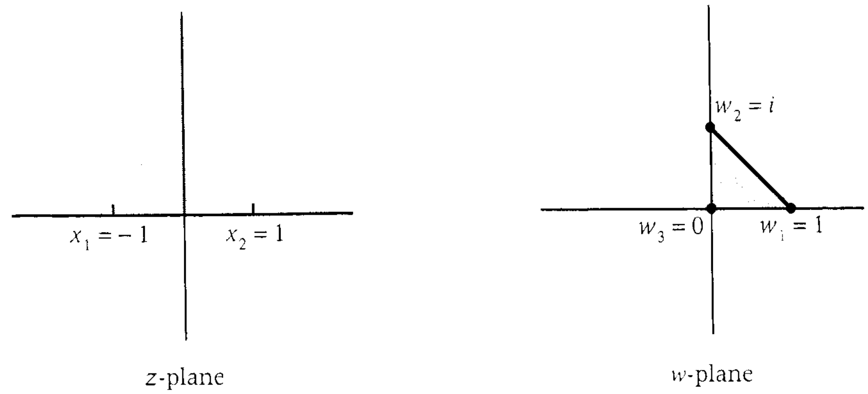

Figure 7.39 Mapping onto a triangle.

where

$$\eta := \int_0^{-1} \left(\zeta^2 - 1\right)^{-3/4} d\zeta \approx 1.85(1 + i)$$

and

$$f(x_2) = f(1) = A \int_0^{1} \left(\zeta^2 - 1\right)^{-3/4} d\zeta + B = -A\eta + B.$$

Setting these equal to w_1 and w_2, respectively, we find

$$A\eta + B = 1,$$
$$-A\eta + B = i.$$

Consequently,

$$A = \frac{1 - i}{2\eta}, \qquad B = \frac{1 + i}{2}. \quad \blacksquare$$

Example 2

Determine a Schwarz-Christoffel transformation that maps the upper half-plane onto the semi-infinite strip $|\operatorname{Re} w| < 1$, $\operatorname{Im} w > 0$ (Fig. 7.40).

Solution. We return to the analysis surrounding Eq. (3) for mapping the real axis onto a polygonal path. To have the upper half-plane map onto the interior of the strip we choose the orientation indicated by the arrows in Fig. 7.40. Left turns of $\pi/2$ radians at w_1 and w_2 can be accommodated by a mapping whose derivative is of the form

$$f'(z) = A\,(z - x_1)^{-1/2}\,(z - x_2)^{-1/2}.$$

Choosing $x_1 = -1$ and $x_2 = 1$ again, we compute

$$f(z) = A \int_0^{z} (\zeta + 1)^{-1/2} (\zeta - 1)^{-1/2}\, d\zeta + B = \frac{A}{i} \int_0^{z} \frac{d\zeta}{\sqrt{1 - \zeta^2}} + B$$
$$= \frac{A}{i} \sin^{-1} z + B.$$

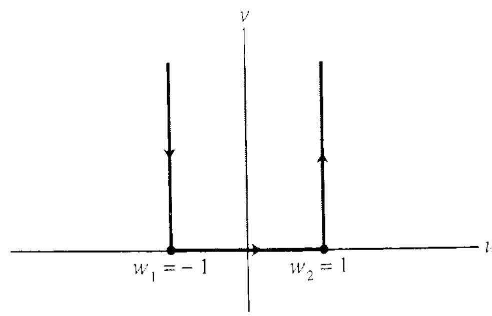

Figure 7.40 Semi-infinite strip for Example 2.

Setting $f(-1) = w_1 = -1$ and $f(1) = w_2 = 1$, we have

$$-iA\sin^{-1}(-1) + B = -1,$$
$$-iA\sin^{-1}(1) + B = 1,$$

which implies that $B = 0$ and $A = 2i/\pi$. Hence

$$f(z) = \frac{2}{\pi}\sin^{-1} z. \quad \blacksquare$$

Example 3

Map the upper half-plane onto the domain consisting of the fourth quadrant plus the strip $0 < v < 1$. (This is a crude model of the continental shelf.)

Solution. The boundary of this domain consists of the line $v = 1$, the negative u-axis, and the negative v-axis. We shall regard this as the limiting form of the polygonal

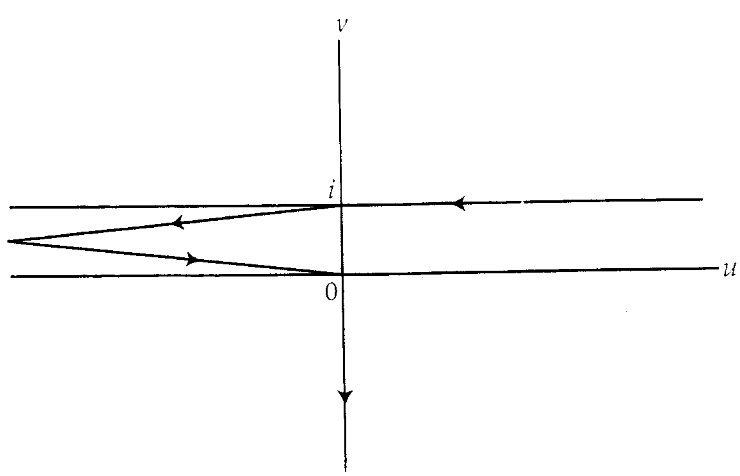

Figure 7.41 Domain for Example 3.

path indicated in Fig. 7.41, again choosing the orientation so that the specified domain lies to the left. A left turn of π radians is called for at the corner "near $w = -\infty$" and a right turn of $\pi/2$ radians occurs at $w = 0$. Selecting $x_1 = -1$ and $x_2 = 1$ as the respective preimages of these points we write, in accordance with Eq. (3),

$$f'(z) = A(z+1)^{-1}(z-1)^{1/2}.$$

Using integral tables, with some labor we arrive at

$$f(z) = Ai\left\{2\sqrt{1-z} + \sqrt{2}\log\frac{\sqrt{1-z}-\sqrt{2}}{\sqrt{1-z}+\sqrt{2}}\right\} + B.$$

The selection of branches is quite involved in this case, so we shall leave it to the industrious reader (Prob. 6) to verify that with the choice

$$\log\zeta = \operatorname{Log}|\zeta| + i\arg\zeta, \qquad -\frac{3}{2}\pi < \arg\zeta \le \frac{\pi}{2},$$

$$\sqrt{\zeta} = e^{(\log\zeta)/2}, \qquad \log\zeta \text{ as above},$$

we find that

$$f(z) = \frac{\sqrt{2}}{\pi}\sqrt{1-z} + \frac{1}{\pi}\log\frac{\sqrt{1-z}-\sqrt{2}}{\sqrt{1-z}+\sqrt{2}} + i$$

satisfies the required conditions

$$\begin{array}{lll}
\operatorname{Re} f(x) \to +\infty, & \operatorname{Im} f(x) \to 1 & \text{as} \quad x \to -\infty, \\
\operatorname{Re} f(x) \to -\infty, & \operatorname{Im} f(x) \to 1 & \text{as} \quad x \to (-1)^-, \\
\operatorname{Re} f(x) \to -\infty, & \operatorname{Im} f(x) \to 0 & \text{as} \quad x \to (-1)^+, \\
f(1) = 0 & & \\
\operatorname{Re} f(x) \to 0, & \operatorname{Im} f(x) \to -\infty & \text{as} \quad x \to +\infty.
\end{array}$$ ■

EXERCISES 7.5

1. Use the techniques in this section to find a conformal map of the upper half-plane onto the whole plane slit along the negative real axis up to the point -1. [HINT: Consider the slit as the limiting form of the wedge indicated in Fig. 7.42.]

2. Use the Schwarz-Christoffel formula to derive the mapping $w = \sqrt{z}$ of the upper half-plane onto the first quadrant.

3. Map the upper half-plane onto the semi-infinite strip $u > 0$, $0 < v < 1$, indicated in Fig. 7.43.

4. Show that the transformation

$$w = \int_0^z \frac{d\zeta}{(1-\zeta^2)^{2/3}}$$

maps the upper half-plane onto the interior of an equilateral triangle.