

Be sure this exam has 6 pages including the cover

The University of British Columbia

MATH 305, Sections 201

Midterm Examination 1, Feb. 10, 2017, 50minutes

Name _____ Signature _____

Student Number _____

This exam consists of 5 questions. No notes. Write your answer in the blank page provided.

Problem	max score	score
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
total	100	

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

(20 points) 1. Find all solutions in the complex plane to the following:

$$\sin(z) = 2i$$

Hint: $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$

Solutions:

Method 1. $\sin z = \sin x \cosh y + i \cos x \sinh y = 2i$

$$\Rightarrow \sin x \cosh y = 0 \Rightarrow \sin x = 0 \Rightarrow x = k\pi$$

$$\cos x \sinh y = 2 \Rightarrow \cos k\pi \sinh y = 2$$

k - even $\Rightarrow \sinh y = 2 \Rightarrow e^{2y} - 4e^y - 1 = 0$

$$e^y = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5} > 0 \Rightarrow$$

$$e^y = 2 + \sqrt{5}$$

$$x = 2m\pi, \quad y = \ln(2 + \sqrt{5})$$

k - odd $\Rightarrow \sinh y = -2 \Rightarrow e^{2y} + 4e^y - 1 = 0$

$$e^y = 2 + \sqrt{5}, \quad y = \ln(\sqrt{5} - 2)$$

$$x = (2m+1)\pi, \quad y = \ln(\sqrt{5} - 2)$$

Method 2. $e^{iz} - e^{-iz} = 4i^2 = -4 \Rightarrow e^{2iz} + 4e^{iz} - 1 = 0$

$$e^{iz} = -2 \pm \sqrt{5}$$

"+" sign: $e^{iz} = -2 + \sqrt{5}, \quad iz = \log(\sqrt{5} - 2) = \ln|\sqrt{5} - 2| + i(2k\pi)$

$$z = -i \ln|\sqrt{5} - 2| + 2k\pi$$

"-" sign: $e^{iz} = -2 - \sqrt{5}, \quad iz = \log(-2 - \sqrt{5}) = \ln|2 + \sqrt{5}| + i(\pi + 2k\pi)$

$$z = -i \ln|2 + \sqrt{5}| + (2k+1)\pi$$

(20 points) 2. Find the image of the following domain

$$\{x > 0, 0 < y < \frac{\pi}{2}\}$$

under the mapping $w = e^z$.

Solution: $w = e^z = e^{x \cos y + i e^{x \sin y}}$

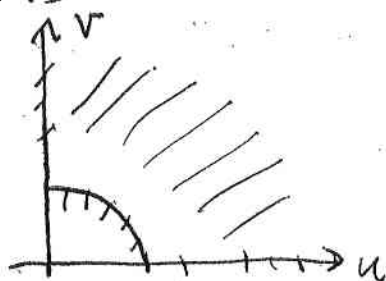
$$u = e^{x \cos y}$$

$$v = e^{x \sin y}$$

$$\left. \begin{array}{l} u = e^{x \cos y} \\ v = e^{x \sin y} \end{array} \right\} \Rightarrow u^2 + v^2 = e^{2x} > 1$$

$$y = \arctan \frac{v}{u} \in (0, \frac{\pi}{2})$$

The image is



$$\{(u, v) \mid u^2 + v^2 > 1, u > 0, v > 0\}$$

(20 points) 3. Find a branch cut for $f(z) = (z(z+2)(z-3))^{\frac{1}{2}}$ so that it is analytic in $C \setminus ((-\infty, -2] \cup [0, 3])$ and $f(-1) = 2$.

Solution: Branch points are $z = 0, -2, 3$
 $(z(z+2)(z-3))^{\frac{1}{2}} = r_1^{\frac{1}{2}} r_2^{\frac{1}{2}} r_3^{\frac{1}{2}} e^{i(\frac{\varphi_1}{2} + \frac{\varphi_2}{2} + \frac{\varphi_3}{2})}$
 $r_1 = |z-3|, r_2 = |z+2|, r_3 = |z|$
 $\varphi_1 = \arg(z-3), \varphi_2 = \arg(z), \varphi_3 = \arg(z+2)$

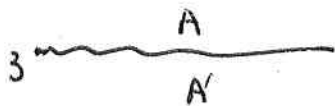


To make it analytic at -1, we take a cut

$$0 < \varphi_1 < 2\pi, \quad 0 < \varphi_2 < 2\pi$$

$$\pi < \varphi_3 < 3\pi$$

First we show that this cut make it $f(z)$ analytic for $z = x > 3$.

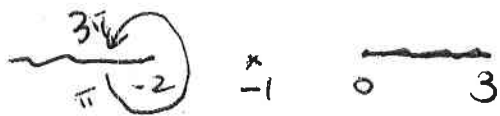


	φ_1	φ_2	φ_3	$\frac{\varphi_1 + \varphi_2 + \varphi_3}{2}$
A	0	0	2π	π
A'	2π	2π	2π	3π

> differ 2π

Second, at $z = -1$

$$\varphi_1 = \pi, \quad \varphi_2 = \pi, \quad \varphi_3 = 2\pi$$



So

$$f(z) = r_1^{\frac{1}{2}} r_2^{\frac{1}{2}} r_3^{\frac{1}{2}} e^{i(\frac{\pi}{2} + \frac{\pi}{2} + \frac{2\pi}{2})} = 2$$

Rmk: If $-\pi < \varphi_3 < \pi$, then $f(-1) = -2$

(20 points) 4. If $u(x, y) = 2xy + x$, find an analytic function $f(z)$ with $\operatorname{Re}(f(z)) = u(x, y)$ and $f(0) = i$.

Solution: Let $f = u + iv$. By Cauchy-Riemann

$$u_x = v_y \Rightarrow v_y = 2y + 1$$

$$\Rightarrow v = y^2 + y + \varphi(x)$$

$$u_y = -v_x \Rightarrow v_x = -u_y = -2x = \varphi'(x)$$

$$\varphi'(x) = -2x, \quad \varphi = -x^2 + a$$

$$f = 2xy + x + i(y^2 + y - x^2 + a)$$

$$f(0) = i \Rightarrow a \cdot i = i \Rightarrow a = 1$$

$$\text{So } f = 2xy + x + i(y^2 + y - x^2 + 1)$$

$$= z - iz^2 + i$$

(20 points) 5. Show that in general $\log(z^2) \neq 2 \log(z)$.

Solution: By definition

$$\log(z^2) = \ln|z^2| + i(\operatorname{Arg}(z^2) + 2k\pi), \quad \text{where}$$

$$-\pi < \operatorname{Arg}(z^2) \leq \pi, \quad k = 0, \pm 1, \dots$$

$$\log z = \ln|z| + i(\operatorname{Arg}(z) + 2m\pi)$$

$$2\log z = 2\ln|z| + 2i(\operatorname{Arg}(z) + 2m\pi)$$

$$= \ln|z^2| + i(2\operatorname{Arg}(z) + 4m\pi), \quad m = 0, \pm 1, \dots$$

Now we choose

$$z = 1, \quad \operatorname{Arg}(z^2) = 0$$

$$k = 1,$$

Then \exists no m such that

$$2\log z = \ln|z^2| + i(2\operatorname{Arg}(z) + 4m\pi) = i(4m\pi) = i \cdot 2\pi$$

So $\log(z^2) \neq 2\log z$ in general