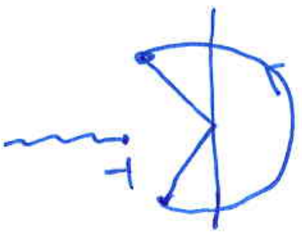
Sol'ns to Midterm Exam 2, MATH305-201, 2016-2017

1. (a) $z = 2e^{i\theta}$, $dz = 2ie^{i\theta}d\theta$

Note that $(\mathrm{Log}(z+1))' = \frac{1}{z+1}$ and $\mathrm{Log}(z+1)$ is analytic on C.



Hence by fundamental theorem of calculus

$$\int_C \frac{1}{z+1}dz = \mathrm{Log}(z(\tfrac{2\pi}{3})+1) - \mathrm{Log}(z(-\tfrac{2\pi}{3})+1)$$

$$= \mathrm{Log}(2e^{i\frac{2\pi}{3}}+1) - \mathrm{Log}(2e^{-i\frac{2\pi}{3}}+1)$$

$$= \mathrm{Log}(\sqrt{3}\,i) - \mathrm{Log}(-\sqrt{3}\,i)$$

$$= \ln|\sqrt{3}| + i\cdot\frac{\pi}{2} - (\ln|\sqrt{3}| - i\frac{\pi}{2})$$

$$= \pi i$$

(b) $dz = 2ie^{i\theta}d\theta$, $\bar{z}^2 = 4e^{-2i\theta}$

$$\int_C \bar{z}^2 dz = 8i\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} e^{-i\theta}d\theta = -8e^{-i\theta}\Big|_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}$$

$$= -8\left(e^{-\frac{2\pi i}{3}} - e^{\frac{2}{3}\pi i}\right)$$

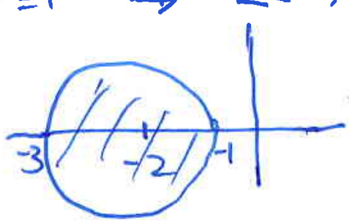
$$= -8\left(-\sin(\tfrac{2\pi}{3})\,i - \sin\tfrac{2}{3}\pi\, i\right)$$

$$= +16\sin\tfrac{2\pi}{3}\, i = +8\sqrt{3}\, i$$

2. (a) $f(z) = \dfrac{e^z}{\sin(z)(z-1)^3}$. It has singularities at

$\sin z = 0$ or $z = 1$ $\Rightarrow$ $z = 0, \pm\pi, \pm 2\pi, \cdots$, or $z = 1$

Now $C: \ |z+2| = 1$



Inside C, $f(z)$ is analytic.

Hence $\int_C f(z)\,dz = 0$.

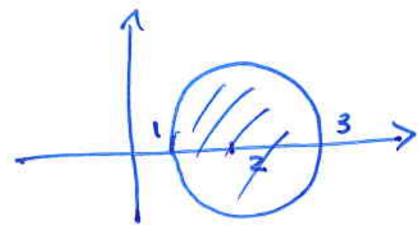
(b) $f(z) = \frac{1}{3i+\sqrt{z}}$

its singularity is $3i+\sqrt{z} = 0 \Rightarrow |\sqrt{z}| = 3$

Now $C: |z-2| = 1$. Inside C, $|z-2| \leq 1 \Rightarrow |z| \leq 3$, $\sqrt{|z|} \leq 3$.

So $f(z)$ has no singularities inside C. Furthermore

$\sqrt{z}$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$



So f is analytic in C

$$\Rightarrow \int_C f(z)\, dz = 0$$

3. By Nyquist's criterion

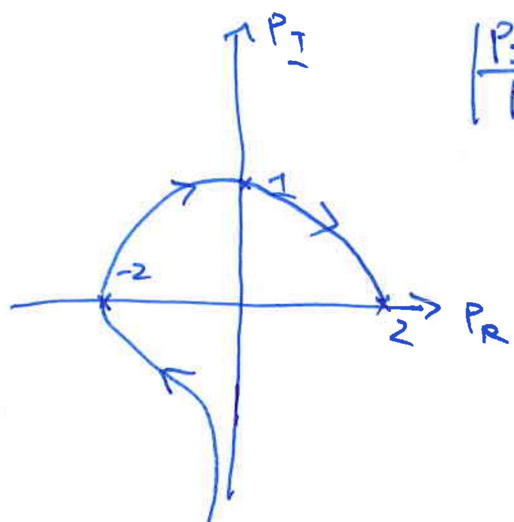
$$N = \frac{1}{2\pi}\left(n\pi + 2\arg(p(z))\Big|_{T_+}\right)$$

T_+: iy, $\quad p(iy) = 2-2y^2+i(2y-y^3)$

$= P_R + i P_I$

$P_R = 0 \Rightarrow y = \pm 1$, $\quad P_I = 0 \Rightarrow y = 0,\ y = \pm\sqrt{2}$

	P_R	P_I
∞	$-\infty$	$-\infty$
$\sqrt{2}$	-2	0
1	0	1
0	2	0



$\left|\frac{P_I}{P_R}\right| \to +\infty$ as $y \to +\infty$

$N = \frac{1}{2\pi}\left(3\pi + 2\times\left(-\frac{3}{2}\pi\right)\right) = 0. \Rightarrow p(z)$ has no roots in $Re(z) \geq 0$

4. (a). $f(z) = (z^2+1)e^{\frac{1}{z}}$

It has an essential singularity at $z=0 \Rightarrow$

$$f(z) = (z^2+1)\left(1+\frac{1}{z}+\frac{1}{2}\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+O\left(\frac{1}{z^4}\right)\right)$$

$$= z^2+z+\frac{1}{2}+\frac{1}{3!}\frac{1}{z}+1+\frac{1}{z}+\frac{1}{2}\frac{1}{z^2}+O\left(\frac{1}{z^3}\right)$$

$$\operatorname{Res}(f;0) = 1+\frac{1}{3!} = \frac{7}{6}$$

So $\displaystyle\int_{|z|=1} f(z)\,dz = 2\pi i\cdot\frac{7}{6} = \frac{7\pi i}{3}$

(b). $\tan z = \dfrac{\sin z}{\cos z}$, $\cos z = 0 \Rightarrow z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$

$z_1 = \frac{\pi}{2}$, $z_2 = -\frac{\pi}{2}$ are inside $|z|\le 2$

$$\operatorname{Res}(\tan z; z_1) = \frac{\sin z_1}{-\sin z_1} = -1$$

$$\operatorname{Res}(\tan z; z_2) = \frac{\sin z_2}{-\sin z_2} = -1$$

$$\int_{|z|=2}\tan z = 2\pi i(-2) = -4\pi i$$

5. $z = e^{i\theta}$, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+z^{-1}}{2}$

$\cos 2\theta = \dfrac{z^2+z^{-2}}{2}$, $dz = ie^{i\theta}d\theta = iz\,d\theta$

$$\int_0^{2\pi}\frac{2\cos 2\theta}{3+2\cos\theta}d\theta = \int_{|z|=1}\frac{z^2+z^{-2}}{3+z+z^{-1}}\frac{1}{iz}dz = \int_{|z|=1}\frac{z^4+1}{z^2(z^2+3z+1)}dz\ \frac{1}{i}$$

$f(z) = \dfrac{z^4+1}{z^2(z^2+3z+1)}$ has singularity $z_1=0$, $z_2 = \dfrac{-3\pm\sqrt{5}}{2} = -\frac{3}{2}+\frac{\sqrt{5}}{2}$, inside $|z|\le 1$.

$$\operatorname{Res}(f;z_1) = \left(\frac{z^4+1}{z^2+3z+1}\right)'\Bigg|_{z=0} = \frac{4z^3}{z^2+3z+1} - \frac{(z^4+1)(2z+3)}{(z^2+3z+1)^2}\Bigg|_{z=0} = 0-3 = -3$$

$$\operatorname{Res}(f;z_2) = \frac{z_2^4+1}{z_2^2(2z_2+3)}$$

$z_2^2 = -3z_2-1 = -3\left(\frac{-3}{2}+\frac{\sqrt{5}}{2}\right)-1 = \frac{7}{2}-\frac{3\sqrt{5}}{2}$

$z_2^{-1} = -\frac{3}{2}-\frac{\sqrt{5}}{2}$, $z_2^{-2} = \frac{7}{2}+\frac{3\sqrt{5}}{2}$

$$= \frac{z_2^2+z_2^{-2}}{2z_2+3} = \frac{\frac{7}{2}+\frac{7}{2}}{2\left(-\frac{3}{2}+\frac{\sqrt{5}}{2}\right)+3} = \frac{7}{\sqrt{5}}$$

$$\int_0^{2\pi}\frac{2\cos 2\theta}{3+2\cos\theta}d\theta = \frac{1}{i}\cdot 2\pi i\left(-3+\frac{7}{\sqrt{5}}\right) = 2\pi\left(-3+\frac{7}{\sqrt{5}}\right)$$