Solutions to Assignment One

1. Sol'n: We use method of characteristics

$$\frac{dx}{2} = \frac{dy}{1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + \xi$$

$$\xi = y - \frac{1}{2}x$$

Hence $u = f(y - \frac{1}{2}x)$

Now $u(x,x) = f(x - \frac{1}{2}x) = f(\frac{x}{2}) = e^{x}$. Let $t = \frac{x}{2}$, $f(t) = e^{2t}$

Hence $u = f(y - \frac{1}{2}x) = e^{2(y - \frac{1}{2}x)} = e^{2y - x}$

2. Sol'n: $\frac{dx}{1} = \frac{dy}{3x^2 y} \Rightarrow \frac{dy}{dx} = 3x^2 y \Rightarrow \frac{dy}{y} = 3x^2\,dx$

$$\Rightarrow \int \frac{dy}{y} = \int 3x^2\,dx \Rightarrow \ln y = x^3 + C$$

$$\Rightarrow y = \xi e^{x^3} \qquad \xi = y e^{-x^3} \Rightarrow u = f(y e^{-x^3})$$

(If $y<0$, we take $\xi<0$)

$u(0,y) = y \Rightarrow u(0,y) = f(y) = y$

Hence $u = f(y e^{-x^3}) = y e^{-x^3}$

The solution is defined everywhere

3. Sol'n: The initial curve can be parametrized by

$(1, \xi)$, $u_0(\xi) = \xi^2$, $0 \le \xi \le 1$

$$\frac{dx}{x} = \frac{dy}{x+y} = \frac{du}{u} = ds$$

$$\begin{cases} \dfrac{dx}{ds} = x, \quad x(0) = 1 & (1) \\[2ex] \dfrac{dy}{ds} = x + y, \quad y(0) = \xi, \quad 0 \le \xi \le 1 & (2) \\[2ex] \dfrac{du}{ds} = u, \quad u(0) = \xi^2 & (3) \end{cases}$$

From (1) $\Rightarrow$ $x = e^s$

Substitute into (2) $\Rightarrow$ $\dfrac{dy}{ds} = y + e^s \Rightarrow y = se^s + Ae^s$

$y(0) = \xi \Rightarrow A = \xi, \quad y = se^s + \xi e^s$

From (3) $\Rightarrow$ $\dfrac{du}{u} = ds \Rightarrow u = ce^s \Rightarrow u = \xi^2 e^s$

Hence

$x = e^s \Rightarrow e^s = x \Rightarrow s = \ln x$

$y = se^s + \xi e^s \Rightarrow \xi = ye^{-s} - s = \dfrac{y}{x} - \ln x$

$u = \xi^2 e^s$

$$u = \left(\frac{y}{x} - \ln x\right)^2 \cdot x = \frac{(y - x\ln x)^2}{x}$$

Now $0 \le \xi \le 1 \Rightarrow 0 \le \dfrac{y}{x} - \ln x \le 1$

The solution is defined in the region where

$$0 \le \frac{y}{x} - \ln x \le 1$$

4. We take care of different boundary conditions.

$$\frac{dt}{1} = \frac{dx}{x+1} = \frac{du}{4u} = ds$$

$x>0,\ u(x,0)=1$: can be parametrized by $(\xi, 0)$, $u_0(\xi)=1$, $\xi>0$

$$\frac{dt}{ds}=1,\ t(0)=0 \quad \Rightarrow \quad t=s$$

$$\frac{dx}{ds} = x+1,\ x(0)=\xi \quad \Rightarrow \frac{dx}{ds} = x+1 \Rightarrow x+1 = ce^{s} \Rightarrow x = -1+(\xi+1)e^{s},\ \xi>0$$

$$\frac{du}{ds} = 4u,\ u(0)=1 \quad \Rightarrow u = e^{4s}$$

Hence the solution is

$$u = e^{4t}$$

in the region $\xi>0 \Rightarrow (x+1)e^{-s}-1=\xi \Rightarrow (x+1)e^{-t}-1>0 \Rightarrow$
$x+1>e^{t}$

$t>0,\ u(0,t)=t$: can be parametrized by $(0,\xi)$, $u_0(\xi)=\xi$, $\xi>0$

$$\frac{dt}{ds}=1,\ t(0)=\xi \quad \Rightarrow t=s+\xi$$

$$\frac{dx}{ds}=x+1,\ x(0)=0 \quad \Rightarrow x+1=ce^{s} \Rightarrow x=-1+e^{s}$$

$$\frac{du}{ds}=4u,\ u(0)=\xi \quad \Rightarrow u=\xi e^{4s}$$

$e^{s}=x+1 \quad s=\ln(x+1), \quad \xi = t-s = t-\ln(x+1)$

$u=(t-\ln(x+1))(x+1)^{4}, \quad \xi = t-\ln(x+1)>0 \Rightarrow x+1<e^{t}$

Hence the sol'n is

$$u(x,t) = \begin{cases} e^{4t}, & \text{when } x+1>e^{t} \\ (t-\ln(x+1))(x+1)^{4}, & \text{when } x+1<e^{t} \end{cases}$$

5. Sol'n: the initial curve can be parametrized by

$(\xi, 2e^{\xi}), \quad u_0(\xi) = 1$

$\frac{dx}{ds} = 1, \quad x_0(0) = \xi \quad \Rightarrow \quad x = \xi + s$

$\frac{dy}{ds} = e^{x}, \quad y(0) = 2e^{\xi} \quad \Rightarrow \quad \frac{dy}{ds} = e^{\xi+s} \Rightarrow y = e^{\xi+s} + C \Rightarrow y = e^{\xi+s} + e^{\xi}$

$\frac{du}{ds} = -u^2, \quad u(0) = 1 \quad \Rightarrow \quad \frac{1}{u} = s + c \Rightarrow u = \frac{1}{s+1}$

Hence $\quad x = \xi + s \quad \Rightarrow s = x - \ln(y - e^{x})$

$y = e^{\xi+s} + e^{\xi} \Rightarrow e^{\xi} = y - e^{x} \Rightarrow \xi = \ln(y - e^{x})$

$$u = \frac{1}{x - \ln(y - e^{x}) + 1}$$

It becomes unbounded when $\quad x - \ln(y - e^{x}) + 1 = 0$

$$\Rightarrow y - e^{x} = e^{x+1} \Rightarrow y = (e-1)e^{x}$$

6. (a). $\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y + c \Rightarrow \xi = \frac{y}{x}$

Let $x' = x$ $\quad u(x,y) = U(x', y')$

$y' = \xi = \frac{y}{x}$

Then $x u_x + y u_y = x U_{x'} = x' U_{x'} = x^2 u = x'^2 U$

$\Rightarrow U_{x'} = x' U \Rightarrow \frac{dU}{U} = x' \, dx'$

$\Rightarrow \ln U = \frac{x'^2}{2} + C(y') \Rightarrow U = f(y') e^{\frac{x'^2}{2}} = f(\frac{y}{x}) e^{\frac{x^2}{2}}$

Then the general sol'n is $u(x,y) = U = f(\frac{y}{x}) e^{\frac{x^2}{2}}$

(b). Let $u = h(x)$ when $y = x$. Then

$$h(x) = f(1)\, e^{\frac{x^2}{2}}.$$

Thus $h(x)$ must be of the form

$$C e^{\frac{x^2}{2}}, \quad C \text{ is a constant.}$$