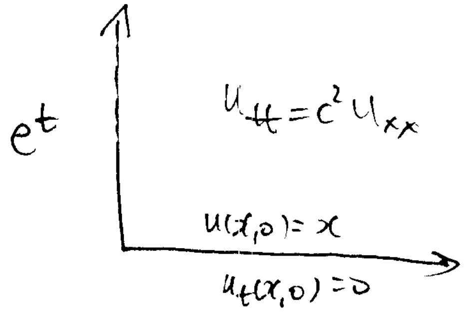
Solutions to Assignment 4 – MATH 400-201

1. We follow the method in Lecture Note 5, page 4.



The general solution of wave equation is

$$u = f(x-ct) + g(x+ct)$$

For IC, $u(x,0) = x$, $u_t(x,0) = 0$, $x > 0$ $\Rightarrow$

$$f(x) + g(x) = x$$

$$-cf'(x) + cg'(x) = 0 \Rightarrow f(x) - g(x) = A$$

Hence $f(x) = \frac{x+A}{2}$, $g(x) = \frac{x-A}{2}$, $x > 0$

For BC, $u(0,t) = e^t$, we have

$$f(-ct) + g(ct) = e^t, \quad t > 0$$

So

$$f(-x) + g(x) = e^{\frac{x}{c}}, \quad x > 0$$

$$f(-x) = e^{\frac{x}{c}} - g(x) = e^{\frac{x}{c}} - \frac{x-A}{2}, \quad x > 0$$

$$f(x) = e^{-\frac{x}{c}} - \frac{-x-A}{2}, \quad x < 0$$

Thus for $x > ct$,

$$u(x,t) = f(x-ct) + g(x+ct) = \frac{x-ct+A}{2} + \frac{x+ct-A}{2}$$

$$= x$$

for $x < ct$

$$f(x-ct) = e^{-\frac{x-ct}{c}} - \frac{-(x-ct)-A}{2}$$

$$u(x,t) = e^{-\frac{x-ct}{c}} - \frac{-(x-ct)-A}{2} + \frac{x+ct-A}{2}$$

$$= e^{-\frac{x-ct}{c}} + x$$

Finally

$$u(x,t) = \begin{cases} x, & \text{for } x > ct \\ e^{t-\frac{x}{c}} + x, & \text{for } x < ct \end{cases}$$

2. We use method of reflection.

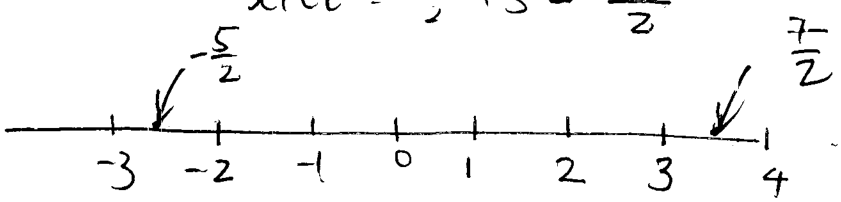
$\phi(x) = 1$, $\psi(x) = 1$

$$\phi_{odd}(x) = \begin{cases} 1, & 0 < x < 1 \\ -1 & -1 < x < 0 \\ \text{periodic with period } 2 \end{cases} , \quad \psi_{odd}(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & -1 < x < 0 \\ \text{periodic with period } 2 \end{cases}$$

$u(x,t) = u(\frac{1}{2}, 3)$ $\quad x = \frac{1}{2}, \; t = 3$

$c = 1$, $\quad x - ct = \frac{1}{2} - 3 = -\frac{5}{2}$

$\quad x + ct = \frac{1}{2} + 3 = \frac{7}{2}$



$$\phi_{odd}(-\tfrac{5}{2}) = \phi_{odd}(-\tfrac{5}{2}+2) = \phi_{odd}(-\tfrac{1}{2}) = -1$$

$$\phi_{odd}(\tfrac{7}{2}) = \phi_{odd}(2+\tfrac{3}{2}) = \phi_{odd}(\tfrac{3}{2}) = \phi_{odd}(\tfrac{3}{2}-2) = \phi_{odd}(-\tfrac{1}{2}) = -1$$

$$\int_{-\frac{5}{2}}^{\frac{7}{2}} \psi_{odd}(s)\,ds = \int_{-\frac{5}{2}}^{-2} + \int_{-2}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{3} + \int_{3}^{\frac{7}{2}}$$

$$= \int_{-\frac{5}{2}}^{-2} + \int_{0}^{1} + \int_{3}^{\frac{7}{2}}$$

$$= \int_{-\frac{1}{2}}^{0} + \int_{0}^{1} + \int_{1}^{\frac{3}{2}}$$

$$= (-1)\times\frac{1}{2} + 1 + (-1)\times\frac{1}{2} = 0$$

Hence

$$u(\tfrac{1}{2},3) = \frac{1}{2}\left[\phi_{odd}(-\tfrac{5}{2}) + \phi_{odd}(\tfrac{7}{2})\right] + \frac{1}{2}\int_{-\frac{5}{2}}^{\frac{7}{2}} \psi_{odd}(s)\,ds$$

$$= \frac{1}{2}\left[-1+(-1)\right] + 0$$

$$= -1$$

3. $f(x,t) = e^x$, $\phi(x) = x$, hence

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\, y\, dy + \int_0^t \left(\int S(x-y,t-s)\, e^y\, dy\right) ds$$

$$\int_{-\infty}^{+\infty} S(x-y,t)\, y\, dy \;\overset{y = x+\sqrt{4kt}\,p}{=\!=}\; \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4k\pi t}}\, e^{-\frac{(y-x)^2}{4kt}}\, y\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-p^2}\left(x + \sqrt{4kt}\, p\right) dp$$

$$= x$$

$$\int_0^t \int S(x-y, t-s)\, e^y\, dy\, ds$$

$$= \int_0^t \frac{1}{\sqrt{4k\pi(t-s)}} \int e^{-\frac{(y-x)}{4k(t-s)}}\, e^y\, dy\, ds$$

$$= \int_0^t \frac{1}{\sqrt{\pi}} \int e^{-p^2} \cdot e^{x+\sqrt{4k(t-s)}\,p}\, dp\, ds$$

$$= \frac{1}{\sqrt{\pi}} \int_0^t \int e^{-(p+\sqrt{k(t-s)})^2} \cdot e^{x+k(t-s)}\, dp\, ds$$

$$= \frac{1}{\sqrt{\pi}} \int_0^t \sqrt{\pi}\, e^{x+k(t-s)}\, ds$$

$$= \int_0^t e^{k(t-s)}\, ds\, e^x = \frac{1}{k} e^x (e^{kt}-1)$$

Hence

$$u(x,t) = x + \frac{1}{k} e^x (e^{kt}-1)$$

4. We use method of extension.

First of all, let

$$u(x,t) = 1 + v(x,t)$$

Then

$$\begin{cases} v_t = k v_{xx} \\ v(x,0) = -1, \quad 0 < x < +\infty \\ v(0,t) = 0 \end{cases}$$

$$\phi(x) = -1, \quad \phi_{odd}(x) = \begin{cases} -1, & 0 < x < +\infty \\ 1, & -\infty < x < 0 \end{cases}$$

So

$$V(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\, \phi_{odd}(y)\, dy$$

$$= \int_{-\infty}^{0} S(x-y,t)\, dy + \int_{0}^{+\infty} S(x-y,t)(-1)\, dy$$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(y-x)^2}{4kt}} dy - \int_{0}^{+\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(y-x)^2}{4kt}} dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{x}{\sqrt{4kt}}} e^{-p^2} dp - \int_{-\frac{x}{\sqrt{4kt}}}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-p^2} dp$$

$$= \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} + \int_{0}^{-\frac{x}{\sqrt{4kt}}} e^{-p^2} dp - \left(\frac{\sqrt{\pi}}{2} + \int_{-\frac{x}{\sqrt{4kt}}}^{0} e^{-p^2} dp \right) \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[-2 \int_{0}^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp \right]$$

5. Let u_1, u_2 be two solutions, and let

$v(x,t) = u_1(x,t) - u_2(x,t)$

Then v satisfies

$$\begin{cases} v_t = k v_{xx}, & 0 < x < l,\ t > 0 \\ v(x,0) = 0 \\ v_x(0,t) - a_0 v(0,t) = 0, & v_x(l,t) + a_1 v(l,t) = 0 \end{cases}$$

Let $E(t) = \frac{1}{2} \int_0^l v^2(x,t)\, dx$

Then $$\frac{dE}{dt} = \int_0^l v v_t \, dx$$

$$= k \int_0^l v v_{xx} \, dx$$

$$= k \left(\int_0^l (v v_x)_x - v_x^2 \, dx \right)$$

$$= k \left(v v_x \Big|_0^l - \int_0^l v_x^2 \, dx \right)$$

$$= k \left[v(l,t) v_x(l,t) - v(0,t) v_x(0,t) - \int_0^l v_x^2 \, dx \right]$$

$$= k \left[-a_1 v^2(l,t) - a_0 v^2(0,t) - \int_0^l v_x^2 \, dx \right]$$

$$\leq 0$$

Hence $E(t) \leq E(0) = \int_0^l v^2(x,0)\, dx = 0$

$E(t) \equiv 0 \Rightarrow v^2(x,t) = 0, \Rightarrow v(x,t) \equiv 0$

So $u_1 \equiv u_2$ #