Solutions to Assignment 5 - MATH400-201

1. Step 1 $u = X(x)T(t)$

$$X'' + \lambda X = 0, \quad X'(0) = X'(\ell) = 0$$

$$T' + k\lambda T = 0$$

Step 2: Solve (EVP)

$$\lambda_0 = 0, \quad X_0(x) = 1, \quad T_0(t) = 1$$

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad X_n(x) = \cos\left(\frac{n\pi}{\ell}x\right), \quad n = 1, 2, \cdots$$

$$T_n(t) = e^{-k\lambda_n t}$$

Step 3: Sum-up

$$u(x,t) = \sum_{n=0}^{+\infty} A_n X_n(x) T_n(t)$$

$$A_n = \frac{\int \phi X_n}{\int X_n^2} \qquad n = 0, 1, 2, \cdots$$

For $n=0$, $\int X_0^2 = \ell$, $A_0 = \frac{1}{\ell}\int_0^\ell \phi(x)\,dx = \frac{1}{\ell}\int_0^\ell x\,dx = \frac{\ell}{2}$

$n = 1, 2, \cdots$, $\int_0^\ell X_n^2 = \frac{\ell}{2}$, $A_n = \frac{2}{\ell}\int_0^\ell x\cos\left(\frac{n\pi}{\ell}x\right)dx$

$$= \frac{2}{\ell}\left[x \sin\frac{n\pi}{\ell}x \cdot \frac{\ell}{n\pi} + \left(\frac{\ell}{n\pi}\right)^2 \cos\frac{n\pi}{\ell}x\right]\Big|_0^\ell$$

$$= \frac{2}{\ell}\left(\frac{\ell}{n\pi}\right)^2 \left[\cos n\pi - 1\right] = \frac{2\ell}{(n\pi)^2}\left[(-$$

Thus

$$u(x,t) = \frac{\ell}{2} + \sum_{n=1}^{+\infty} \frac{2\ell}{(n\pi)^2}\left[(-1)^n - 1\right] e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \cos\left(\frac{n\pi}{\ell}x\right)$$

As $n \to +\infty$, $u(x,t) \longrightarrow \frac{\ell}{2}$.

2 (a). Since $a_0 = +\infty$, $a_\ell = 0$, $\lambda > 0$

Assume that $\lambda = \beta^2 > 0$, $\beta > 0$.

$$X(x) = C_1 \cos \beta x + C_2 \sin \beta x$$

$$X(0) = 0 \Rightarrow C_1$$

$$X'(\ell) = 0 \Rightarrow C_2 \beta \cos \beta \ell = 0 \Rightarrow \cos \beta \ell = 0$$

$$\beta \ell = \frac{(2n-1)}{2} \pi, \quad n = 1, 2, \cdots$$

$$\lambda_n = \left(\frac{(2n-1)}{2\ell} \pi \right)^2, \quad n = 1, 2, \cdots$$

$$X_n = \sin \left(\frac{2n-1}{2\ell} \pi x \right)$$

(b). Using the method of separation of variables,

$$u(x,t) = \sum_{n=1}^{+\infty} \left(A_n \cos \sqrt{\lambda_n} c t + B_n \sin \sqrt{\lambda_n} c t \right) \sin \left(\frac{2n-1}{2\ell} \pi x \right)$$

$$u(x,0) = 0 \Rightarrow \sum_{n=1}^{+\infty} A_n \sin \left(\frac{2n-1}{2\ell} \pi x \right) = 0 \Rightarrow A_n = 0$$

$$u_t(x,0) = 2 \sin \frac{3\pi}{2\ell} x \Rightarrow \sum_{n=1}^{+\infty} \sqrt{\lambda_n} c B_n \sin \left(\frac{2n-1}{2\ell} \pi x \right) = 2 \sin \frac{3\pi}{2\ell} x$$

$$\Rightarrow c \sqrt{\lambda_3} B_3 = 2, \quad B_n \neq 0 \text{ if } n \neq 3$$

$$B_3 = \frac{2}{\sqrt{\lambda_3} c} = \frac{2}{\frac{3\pi}{2\ell} c} = \frac{4\ell}{3\pi c}$$

So $u(x,t) = \frac{4\ell}{3\pi c} \sin \left(\frac{3\pi}{2\ell} c t \right) \sin \left(\frac{3\pi}{2\ell} \pi x \right)$

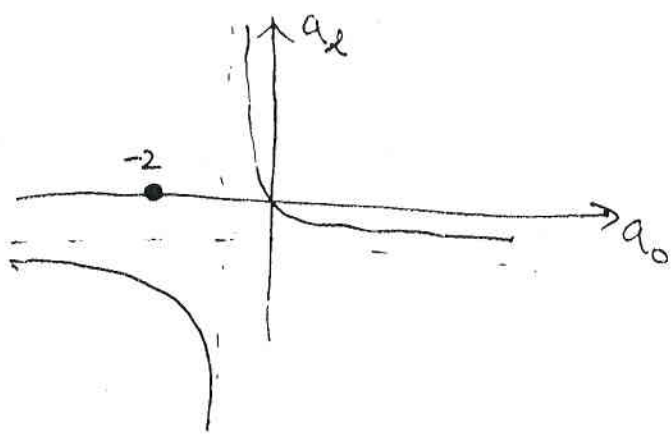
(a). Step1. $u = X(x)T(t)$

$$\Rightarrow \begin{cases} X'' + \lambda X = 0, & 0 < x < 1 \\ X'(0) + 2X(0) = 0, & X'(1) = 0 \end{cases} \qquad T'(t) + \lambda T = 0$$

Step 2 Solve (EVP).

Now $a_0 = -2$, $a_l = 0$, $l = 1$

Hyperbola: $a_0 + a_l + a_0 a_l = 0$



So $\exists$ a unique negative eigenvalue $\lambda_{-1} = -\gamma^2$

all other eigenvalues are positive.

By the formula in Lecture Note 9.5.

$\lambda_{-1} = -\gamma^2$, $\tanh \gamma l = \tanh \gamma = \frac{2}{\gamma}$, $X_{-1} = \cosh \gamma x - \frac{2}{\gamma} \sinh \gamma x$

$\lambda_n = \beta_n^2$, $\tan \beta_n = -\frac{2}{\beta_n}$, $X_n = \cos \beta_n x - \frac{2}{\beta_n} \sin \beta_n x$

Step3. $$u(x,t) = A_{-1} e^{\gamma^2 k t} X_{-1}(x) + \sum_{n=1}^{+\infty} A_n e^{-k\beta_n^2 t} \left(\cos \beta_n x - \frac{2}{\beta_n} \sin \beta_n x\right)$$

$$A_{-1} = \frac{\int_0^l \phi(x) X_{-1}(x)\,dx}{\int X_{-1}^2}, \quad A_n = \frac{\int_0^l \phi(x) X_n(x)\,dx}{\int X_n^2}, \quad n = 1, 2, \ldots$$

(b) If $n \geq 1$, $e^{-k\beta_n t} \to 0$ as $t \to +\infty$

Hence if we want $u(x)$ to be bounded, then necessarily

$$A_{-1} = 0 \Rightarrow \int_0^l \phi(x)\left(\cosh \gamma x - \frac{2}{\gamma}\sinh(\gamma x)\right) = 0$$

So under the condition that

$$\int_0^l \phi(x)\left(\cosh \gamma x - \frac{2}{\gamma}\sinh \gamma x\right) = 0$$

we have

$$u(x,t) \to 0 \quad \text{as } t \to +\infty$$

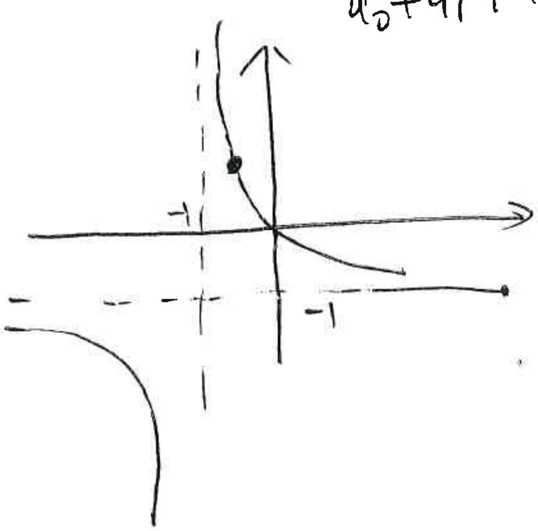
4. We use the formula in Lecture 9.5.

(a) $X'(0) + \frac{1}{2}X(0) = 0 \Rightarrow a_0 = -\frac{1}{2}$

$X'(1) + X(1) = 0 \Rightarrow a_1 = 1$,
$l = 1$

The hyperbola is

$$a_0 + a_1 + a_0 a_1 = 0$$



(a_0, a_1) lies on Region II

So $\exists 1$ ~~negative~~ zero eigenvalue

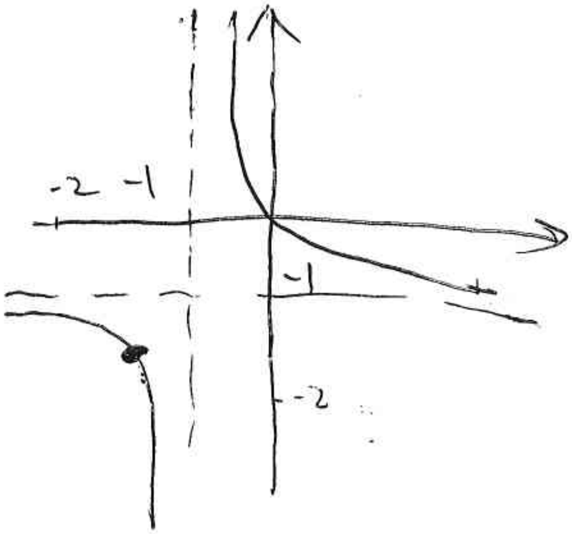
$\lambda_0 = 0 < \lambda_1 = \beta_1^2 < \cdots$

$\lambda_0 = 0$, $X_0(x) = 1 - \frac{x}{2}$

$\lambda_n = \beta_n^2$, $\tan \beta_n = \dfrac{\frac{1}{2}\beta_n}{\beta_n^2 + \frac{1}{2}} = \dfrac{\beta_n}{2\beta_n^2 + 1}$

(b) $l=1,\ a_0=-2,\ a_1=-2$

$$a_0+a_1+a_0a_1 l=0$$



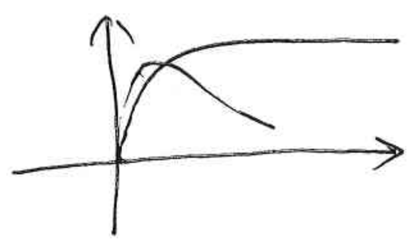
$(a_0, a_1) \in$ Region IV

so $\exists$ one negative eigenvalue and one zero eigenvalue

$$\lambda_{-1}=-\gamma_1^2<\lambda_0=0<\lambda_1=\beta_1^2<\cdots<\lambda_n=\beta_n^2<\cdots$$

Equation $\lambda_{-1}=-\gamma_1^2$

$$\tanh\gamma_1=\frac{4\gamma_1}{\gamma_1^2+4}$$



$$X_{-1}(x)=\cosh\gamma_1 x-\frac{2}{\gamma_1}\sinh\gamma_1 x$$

Equation $\lambda_0=0$, $\quad X_0(x)=1+2x$

Equation for positive $\lambda=\beta^2$

$$\tan\beta=\frac{4\beta}{4-\beta^2}$$

5 (a) $\mu X''+\mu x X'+\lambda\mu X=0 \quad \Rightarrow$

$pX''+p'X'-qX+\lambda wX=0$

$$\left(e^{\frac{x^2}{2}}X'\right)'+\lambda e^{\frac{x^2}{2}}X=0$$

$\left.\begin{array}{l} p=\mu \\ p'=\mu x \end{array}\right\} \Rightarrow p=e^{\frac{x^2}{2}}$

$q=0 \Rightarrow q=0$

$\mu=w \Rightarrow w=e^{\frac{x^2}{2}}$

(b) $\mu X'' + \mu \cdot \frac{1}{x} X' + \lambda \mu X = 0$

$p X'' + p' X + \lambda w X = 0$

$$\left.\begin{array}{l} \mu = p \\ \frac{\mu}{x} = p' \end{array}\right\} \Rightarrow \frac{p'}{p} = \frac{1}{x} \Rightarrow p = x$$

$\mu = w \Rightarrow \mu = w$

$(x X')' + \lambda x X = 0.$

(c) $\mu X'' + \mu \left(\frac{1}{x} - x\right) X' + \lambda \mu X = 0$

$p X'' + p' X + \lambda w X = 0$

$$\left.\begin{array}{l} \mu = p \\ \mu\left(\frac{1}{x} - x\right) = p' \end{array}\right\} \Rightarrow \frac{p'}{p} = \frac{1}{x} - x \Rightarrow \ln p = -\frac{x^2}{2} + \ln x$$

$\Rightarrow p = x e^{-\frac{x^2}{2}}$

$\mu = w$

$w = \mu = p = x e^{-\frac{x^2}{2}}$

$$\left(x e^{-\frac{x^2}{2}} X'\right)' + \lambda \left(x e^{-\frac{x^2}{2}}\right) X = 0.$$