MATH400-201 Homework Assignment 6 (Due Date: March 31, 2016)

1. Put the following two problems in standard Sturm-Liouville form, identify the weight function $w(x)$, and calculate the eigenvalues and eigenfunctions. Write down the general formula for the expansion of a general function $f(x)$ in terms of the eigenfunctions.

$$(a) \quad x^2X'' + 3xX' + \lambda X = 0, 1 < x < 2, \quad X(1) = 0, \quad X(2) = 0$$

$$(b) \quad X'' - 2X' + \lambda X = 0, 0 < x < 1; \quad X(0) = 0, \quad X(1) = 0$$

2. Solve the following wave equation

$$\begin{cases} u_{tt} = u_{xx} - 2u_x, \ 0 < x < 1 \\ u(x,0) = 1, u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

3. Use the method of separation of variables to solve

$$\begin{cases} u_t = u_{xx} + e^t \sin(3x), \ 0 < x < \pi \\ u(x,0) = \sin(5x) \\ u(0,t) = t, \ u(\pi,t) = 0 \end{cases}$$

4. (a) Use the method of separation of variables to solve

$$\begin{cases} u_{xx} + u_{yy} = 0, 0 < x < \pi, \ 0 < y < \pi, \\ u_x(0,y) = u(\pi,y) = u(x,\pi) = 0, \\ u(x,0) = \cos^2(x) \end{cases}$$

(b) Use Energy Method to show the solutions to (a) are unique.

5. The Laplace equation in the polar coordinates is

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r + \frac{\Delta_{S^{N-1}} u}{r^2}$$

where N is the dimension and $\Delta_{S^{N-1}}$ is the Laplace-Betrami operator on S^{N-1}.

When $N = 2$, it becomes

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{u_{\theta\theta}}{r^2}$$

(a) Find all radially symmetric solutions to

$$\Delta u = 0, \quad u = u(r)$$

in R^N.

(b) Solve two-dimensional Laplace equation

$$\Delta u = 0, \ 0 < r < 1, 0 \le \theta < 2\pi$$
$$u(1,\theta) = 1 + 2\cos(\theta) + 3\sin(2\theta)$$