Solutions to Assignment 6, MATH400-201

1 (a). $\mu x^2 X'' + 3\mu x X' + \lambda\mu X = 0$

$p X'' + p' X' q X + \lambda w X = 0$

$\Rightarrow \quad \left.\begin{array}{l} \mu x^2 = p \\ 3\mu x = p' \end{array}\right\} \Rightarrow \quad \frac{p'}{p} = \frac{3x}{x^2} = \frac{3}{x} \Rightarrow p = x^3, \ \mu = x$

$0 = q$

$\mu = w \qquad \Rightarrow \quad w = x$

S-L: $\quad (x^3 X')' + \lambda x X = 0$

The weight function is $w(x) = x$.

Eigenvalues: By Property (iv), $\lambda > 0$.. Since this is a Euler typer, try $X = x^r \Rightarrow \quad r(r-1) + 3r + \lambda = 0$

$r^2 + 2r + \lambda = 0$

$(r+1)^2 + \lambda - 1 = 0 \qquad r = -1 \pm \sqrt{1-\lambda}$

Case 1 $\lambda < 1$. $\quad r_1 = -1 + \sqrt{1-\lambda}, \quad r_2 = -1 - \sqrt{1-\lambda}$

$X = c_1 x^{r_1} + c_2 x^{r_2}$

$\left.\begin{array}{ll} X(1) = 0 \Rightarrow & c_1 + c_2 = 0 \\ X(2) = 0 \Rightarrow & c_1 2^{r_1} + c_2 2^{r_2} = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0.$

Case 2. $\lambda = 1$, $\quad r_1 = r_2 = -1, \quad X = c_1 x^{-1} + c_2 x^{-1} \ln x.$

$X(1) = 0 \Rightarrow \quad c_1 = 0, \qquad X(2) = 0 \Rightarrow c_2 = 0.$

Case 3. $\lambda > 1$ Then

$$X = c_1 x^{-1} \cos(\sqrt{\lambda - 1} \ln x) + c_2 x^{-1} \sin(\sqrt{\lambda - 1} \ln x)$$

$$X(1) = 0 \Rightarrow c_1 = 0$$

$$X(2) = 0 \Rightarrow c_2 2^{-1} \sin(\sqrt{\lambda - 1} \ln 2) = 0 \Rightarrow$$

$$\sqrt{\lambda - 1} \ln 2 = n\pi \Rightarrow \lambda = 1 + \left(\frac{n\pi}{\ln 2}\right)^2, \quad n = 1, 2, \cdots$$

Eigenvalues are

$$\lambda_n = 1 + \left(\frac{n\pi}{\ln 2}\right)^2$$

Eigenfunctions are

$$X_n = x^{-1} \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$$

Expansion

$$f(x) = \sum_{n=1}^{+\infty} a_n X_n(x), \quad \text{where}$$

$$a_n = \frac{\int_1^2 X_n f \, w \, dx}{\int_1^2 X_n^2 \, w \, dx} = \frac{\int_1^2 \cancel{x} \sin\left(\frac{n\pi}{\ln 2} \ln x\right) f(x) \, dx}{\int_1^2 x^{-1} \sin^2\left(\frac{n\pi}{\ln 2} \ln x\right) dx}$$

Note that $\int_1^2 x^{-1} \sin^2\left(\frac{n\pi}{\ln 2} \ln x\right) dx = \int_0^{\ln 2} \sin^2\left(\frac{n\pi}{\ln 2} t\right) dt = \frac{\ln 2}{2}$

So $a_n = \frac{2}{\ln 2} \int_1^2 \sin\left(\frac{n\pi}{\ln 2} \ln x\right) f(x) \, dx$

1(b). $\mu X'' - 2\mu X' + \lambda \mu X = 0$

$pX'' + p'X' + \lambda w X = 0$

$$\left.\begin{array}{l}\mu = p \\ -2\mu = p'\end{array}\right\} \Rightarrow \frac{p'}{p} = -2 \Rightarrow p = e^{-2x}$$

$$\mu = w \quad \Rightarrow \quad w = e^{-2x}.$$

S-L: $(e^{-2x} X')' + \lambda e^{-2x} X = 0$

weight: $w = e^{-2x}$

Eigenvalues: $\lambda = \beta^2 > 0$. Let $X = e^{rx}$

$$r^2 - 2r + \lambda = 0$$

$$(r-1)^2 + \lambda - 1 = 0$$

$$r = 1 \pm \sqrt{1-\lambda}.$$

Case1: $\lambda < 1$. $r_1 = 1 + \sqrt{1-\lambda}$, $r_2 = 1 - \sqrt{1-\lambda}$

$$X = A e^{r_1 x} + B e^{r_2 x}$$

$$X(0) = 0, \ X(1) = 0 \Rightarrow \left.\begin{cases} A + B = 0 \\ Ae^{r_1} + Be^{r_2} = 0 \end{cases}\right\} \Rightarrow A = B = 0$$

Case2: $\lambda = 1$, $r_1 = r_2 = 1$,

$$X = Ae^{x} + Be^{x} x.$$

$$X(0) = 0, \ X(1) = 0 \Rightarrow \begin{cases} A = 0 \\ B = 0. \end{cases}$$

Case3. $\lambda > 1$. $r = 1 \pm \sqrt{\lambda - 1}\, i$

$$X(x) = A e^{x} \cos(\sqrt{\lambda-1}\, x) + B e^{x} \sin(\sqrt{\lambda-1}\, x).$$

$X(0)=0 \Rightarrow A=0$

$X(1)=0 \Rightarrow \sin(\sqrt{\lambda-1})=0 \Rightarrow \sqrt{\lambda-1}=n\pi, \quad \lambda=1+(n\pi)^2$

Eigenvalues are

$$\lambda_n = 1+(n\pi)^2, \quad n=1,2,3\cdots$$

Eigenfunctions are

$$X_n = e^{x} \sin(n\pi x)$$

Expansion:

$$f(x) = \sum_{n=1}^{+\infty} a_n X_n$$

$$a_n = \frac{\int_0^1 X_n f w}{\int_0^1 X_n^2 w} = \frac{\int_0^1 \sin(n\pi x) f(x) e^{-x}\, dx}{\int_0^1 e^{2x} \sin^2(n\pi x)\, e^{-2x}\, dx}$$

$$= \frac{1}{2} \int_0^1 \sin(n\pi x) f(x) e^{-x}\, dx.$$

2. We use method of separation of variables

<u>Step 1</u>: $X''-2X+\lambda X=0, \quad X(0)=X(1)=0$

$T''=\lambda T$

<u>Step 2</u>: By 1b), $\lambda_n = 1+(n\pi)^2, \quad X_n = e^{x} \sin(n\pi x).$

So $T = c_1 \cos\sqrt{\lambda_n}\, t + c_2 \sin\sqrt{\lambda_n}\, t$

Step3: Sum-up

$$u(x,t) = \sum_{n=1}^{+\infty} \left(a_n \cos\sqrt{\lambda_n}\, t + b_n \sin\sqrt{\lambda_n}\, t\right) e^{x} \sin(n\pi x)$$

$$u(x,0) = 1 \Rightarrow 1 = \sum_{n=1}^{+\infty} a_n e^{x} \sin n\pi x .$$

By 1b)

$$a_n = \frac{1}{2} \int_0^1 \sin(n\pi x)\, e^{-x}\, dx$$

$$= \frac{1}{2} \left\{ e^{-x} \left(\frac{n\pi}{(n\pi)^2+1} \cos n\pi x + \frac{1}{(n\pi)^2+1} \sin n\pi x \right) \right|_0^1$$

$$= \frac{1}{2} \left[\frac{(e^{-1}\cos n\pi - 1)\, n\pi}{(n\pi)^2+1} \right]$$

$$u_t(x,0) = 0 \Rightarrow 0 = \sum_{n=1}^{+\infty} \left(\sqrt{\lambda_n}\, b_n \cancel{\cos\sqrt{\lambda_n}}\right) e^{x} \sin(n\pi x)$$

$$\Rightarrow b_n = 0$$

So $$u(x,t) = \sum_{n=1}^{+\infty} \frac{1}{2} \frac{(e^{-1}\cos n\pi - 1)\, n\pi}{(n\pi)^2+1} \cos\sqrt{\lambda_n}\, t \; e^{x} \sin(n\pi x)$$

3. Let $u(x,t) = \sum_{n=1}^{+\infty} u_n(t) \sin(n\cancel{t})$

Then $u_t = \sum_{n=1}^{+\infty} u_n'(t) \sin(nt)$

$$e^{t} \sin 3x = \sum f_n(t) \sin(nx) \Rightarrow \quad f_3(t) = e^{t}, \quad f_n(t) = 0,\ n \neq 3$$

$u(x,0) = \sin 5x$

$\Rightarrow \; u_5(0) = 1, \quad u_n(0) = 0 \quad$ for all $n \neq 5$.

$h(t) = t, \; j(t) = 0$

$$\frac{du_n}{dt} = -n^2 u_n(t) - \frac{2n}{\pi}\left[(-1)^n j(t) - h(t)\right] + f_n(t)$$

$$= -n^2 u_n(t) + \frac{2n}{\pi} t + f_n(t).$$

For $n=3$, we have

$$\begin{cases} u_n'(t) = -n^2 u_n(t) + \frac{2n}{\pi} t + e^t \\ u_n(0) = 0 \end{cases}$$

$$u_n(t) = \int_0^t e^{-n^2(t-s)}\left[\frac{2n}{\pi} s + e^s\right] ds$$

$$= \frac{2}{n\pi}\left[t - \frac{1}{n^2} + \frac{1}{n^2} e^{-n^2 t}\right] + \frac{1}{n^2+1}\left(e^t - e^{-n^2 t}\right)$$

For $n=5$, we have

$$\begin{cases} u_n'(t) = -n^2 u_n(t) + \frac{2n}{\pi} t \\ u_n(0) = 1 \end{cases} \Rightarrow u_n(t) = \int_0^t e^{-n^2(t-s)}\left[\frac{2n}{\pi} s\right] ds + e^{-n^2 t}$$

$$= \frac{2}{n\pi}\left[t - \frac{1}{n^2} + \frac{1}{n^2} e^{-n^2 t}\right] + e^{-n^2 t}$$

For $n \neq 3, 5$, we have

$$\begin{cases} u_n'(t) = -n^2 u_n(t) + \frac{2n}{\pi} t \\ u_n(0) = 0 \end{cases} \Rightarrow u_n(t) = \int_0^t e^{-n^2(t-s)} \left(\frac{2n}{\pi}\right) s \, ds$$

$$= \frac{2}{n\pi}\left[t - \frac{1}{n^2} + \frac{1}{n^2} e^{-n^2 t}\right]$$

4(a) Step 1:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$X'(0) = X(\pi) = 0, \quad Y(\pi) = 0$$

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(\pi) = 0$$

$$Y'' - \lambda Y = 0, \quad Y(\pi) = 0$$

Step 2.

$$\lambda_n = \left(n - \tfrac{1}{2}\right)^2, \quad n = 1, 2, \cdots$$

$$X_n = \cos\left(n - \tfrac{1}{2}\right)x.$$

$$Y'' - \left(n - \tfrac{1}{2}\right)^2 Y = 0, \quad Y(\pi) = 0$$

$$Y = \sinh\left(\left(n - \tfrac{1}{2}\right)(\pi - y)\right)$$

Step 3. Sum-up

$$u(x,y) = \sum_{n=1}^{+\infty} a_n \cos\left(n - \tfrac{1}{2}\right)x \,\sinh\left(\left(n - \tfrac{1}{2}\right)(\pi - y)\right)$$

$$u(x,0) = \cos^2 x = \sum_{n=1}^{+\infty} a_n \sinh\left(\left(n - \tfrac{1}{2}\right)\pi\right) \cos\left(n - \tfrac{1}{2}\right)x.$$

$$\sinh\left(n - \tfrac{1}{2}\right)\pi \; a_n = \frac{\int_0^\pi \cos^2 x \cos\left(n - \frac{1}{2}\right)x}{\int_0^\pi \cos^2\left(n - \frac{1}{2}\right)x} = \frac{2}{\pi} \cdot \int_0^\pi \frac{1 + \cos 2x}{2} \cos\left(n - \tfrac{1}{2}\right)x \, dx$$

$$= \frac{1}{\pi}\left(\frac{1}{n - \frac{1}{2}} \sin\left(n - \tfrac{1}{2}\right)\pi + \frac{1}{2}\frac{1}{n + \frac{3}{2}} \sin\left(n + \tfrac{3}{2}\right)\pi + \frac{1}{2}\frac{1}{n - \frac{5}{2}} \sin\left(n - \tfrac{5}{2}\right)\pi\right)$$

$$= \frac{1}{\pi}\left(\frac{2(-1)^{n-1}}{2n-1} + \frac{(-1)^n}{2n+3} + \frac{(-1)^{n-1}}{2n-5}\right).$$

4(b). Let u_1, u_2 be two solutions. Let

$$v(x,y) = u_1(x,y) - u_2(x,y)$$

Then v satisfies

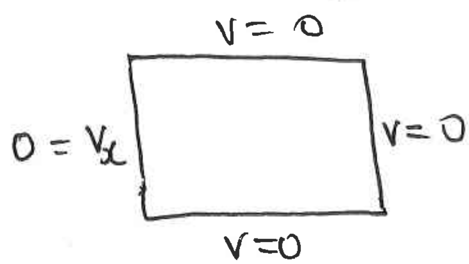
$$\begin{cases} v_{xx} + v_{yy} = 0, & 0<x<\pi, \quad 0<y<\pi \\ v_x(0,y) = v(\pi,y) = v(x,\pi) = 0 \\ v(x,0) = 0 \end{cases}$$

Now we compute, Letting $D = (0,\pi)\times(0,\pi)$

$$0 = \int_0^\pi\int_0^\pi (\Delta v)v = -\int_0^\pi\int_0^\pi |\nabla v|^2 + \int_0^\pi\int_0^\pi \nabla(v\nabla v)$$

$$= \int_{\partial D} v\frac{\partial v}{\partial n} - \int_0^\pi\int_0^\pi |\nabla v|^2$$

For each boundary, either $v=0$ or $\frac{\partial v}{\partial n} = 0$



Hence $\iint_D |\nabla v|^2 = 0$

$\Rightarrow v \equiv \text{Constant}$

$\Rightarrow v = 0$. #

5 a). Let $u = u(r)$. $\Delta u = u_{rr} + \frac{N-1}{r}u_r = 0$

This is Euler type so

$$u = r^{\alpha}.$$

$$\alpha(\alpha-1) + (N-1)\alpha = 0 \Rightarrow \alpha = 0 \text{ or } \alpha = 2-N$$

So the solutions ar

$$u = A + B r^{2-N}$$

(b). We use the formula

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi + \sum_{n=1}^{+\infty}\left(\frac{r}{a}\right)^n\left[\frac{1}{\pi}\int_0^{2\pi} h(\phi)\cos n\phi\,d\phi \cos n\theta + \frac{1}{\pi}\int_0^{2\pi} h(\phi)\sin n\phi\,d\phi \sin n\theta\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi + r\frac{1}{\pi}\int_0^{\pi} h(\phi)\cos\phi\,d\phi\cos\theta + r^2\frac{1}{\pi}\int_0^{2\pi} h(\phi)\sin 2\phi\,d\phi \sin 2\theta$$

$$h = 1 + 2\cos\theta + 3\sin(2\theta)$$

so $\int_0^{2\pi} h(\phi)\,d\phi = 2\pi$, $\int_0^{2\pi} h(\phi)\cos\phi\,d\phi = 2\pi$

$\int_0^{2\pi} h(\phi)\sin 2\phi\,d\phi = 3\pi.$

So

$$u(r,\theta) = 1 + 2r\cos\theta + 3r^2\sin 2\theta.$$