Solutions to Assignment 7, MATH400-201

1 (a). Using polar coordinate

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{u_{\theta\theta}}{r^2} = 0, & r > 1,\ 0 \le \theta < 2\pi \\ u(1, \theta) = \cos\theta - 2\sin^2\theta. \\ u \text{ is bounded} \end{cases}$$

By the method of separation of variables $\Rightarrow$ $h(\theta) := \cos\theta - (1 - \cos 2\theta)$

$$u(r, \theta) = a_0 + \sum_{n=1}^{+\infty} r^{-n} (a_n \cos n\theta + b_n \sin n\theta)$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\phi)\, d\phi = \frac{1}{2\pi} \int_0^{2\pi} (\cos\theta - 2\sin^2\theta)\, d\theta = -1.$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} h(\phi) \cos n\phi\, d\phi = \begin{cases} 1 & n=1 \\ 1 & n=2 \end{cases}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} h(\phi) \sin n\phi\, d\phi = 0, \quad \forall n \ge 1$$

So

$$u(r, \theta) = -1 + r^{-1} \cos\theta + r^{-2} \cos 2\theta$$

$$u(x, y) = -1 + \frac{x}{x^2+y^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

(b). Suppose $\exists$ two sol'ns, u_1, u_2. Let $v = u_1 - u_2$. Then

$$\Delta v = 0, \quad x^2+y^2 > 1$$
$$v = 0, \quad x^2+y^2 = 1$$
$$v \text{ is bounded}$$

Let $w = v - \varepsilon \log r$. Then

$$\Delta w = 0, \quad x^2+y^2 > 1$$
$$w = 0, \quad x^2+y^2 = 1$$
$$w = v - \varepsilon \log R, \quad x^2+y^2 = R$$

By Maximum Principle

$$v - \varepsilon \log r \le \max\{0, v - \varepsilon \log R\} \le 0, \text{ for } R \text{ large}$$

$v \le \varepsilon \log r$, $\forall r \ge 1$ letting $\varepsilon \to 0 \Rightarrow v \le 0$.

Similarly $-v \le 0 \Rightarrow v \ge 0$. Hence $v \equiv 0$.

2. Use the method of separation of variables

Step 1 $\quad R'' + \frac{1}{r}R' - \frac{\lambda}{r^2}R = 0.$

$\Theta'' + \lambda\Theta = 0, \quad \Theta'(0) = \Theta'(\frac{\pi}{4}) = 0,$

Step 2 $\quad \lambda = \left(\frac{n\pi}{\frac{\pi}{4}}\right)^2 = (4n)^2, \ n = 0, 1, 2, \cdots \qquad \Theta = \cos(4n\theta).$

$R = a + b\log r, \quad n = 0$

$R = ar^{4n} + br^{-4n}, \quad n \geq 1.$

Step 3. $\quad u(r,\theta) = a_0 + b_0\log r + \sum_{n=1}^{+\infty}(a_n r^{4n} + b_n r^{-4n})\cos(4n\theta)$

Now. $\quad u(1,\theta) = 2\cos^2 2\theta = \cos 4\theta + 1$

$\cos 4\theta + 1 = a_0 + b_0\log r + \sum_{n=1}^{+\infty}(a_n r^{4n} + b_n r^{-4n})\cos 4n\theta.$

$\Rightarrow \quad a_0 = 1, \quad a_1 + b_1 = 1, \qquad a_n + b_n = 0, \quad n \geq 2.$

$u(2,\theta) = 1 \quad \Rightarrow$

$1 = a_0 + b_0\log 2 + \sum_{n=1}^{+\infty}(a_n 2^{4n} + b_n 2^{-4n})\cos 4n\theta$

$\Rightarrow \quad a_0 + b_0\log 2 = 1, \quad a_1 2^4 + b_1 2^{-4} = 0, \quad a_n 2^{4n} + b_n 2^{-4n} = 0, \ n \geq 2$

so $\quad a_0 = 1, \quad b_0 = 0, \quad a_1 = \frac{1}{1-2^8}, \quad b_1 = \frac{2^8}{2^8-1}.$

$u(r,\theta) = 1 + \left(\frac{1}{1-2^8}r^4 + \frac{2^8}{2^8-1}r^{-4}\right)\cos 4\theta.$

3. We may assume that u is radial. Hence.

$$u = \sum_{m=1}^{+\infty} J_0\left(\frac{z_{m,0}}{a} r\right)\left(a_n \cos \frac{z_{m,0}}{a} ct + b_n \sin \frac{z_{m,0}}{a} ct\right).$$

$$u(r,0) = \phi(r) = \sum_{m=1}^{+\infty} J_0\left(\frac{z_{m,0}}{a} r\right) a_n \Rightarrow$$

$$a_n = \frac{\int_0^a J_0\left(\frac{z_{m,0}}{a} r\right)\phi(r)\, r\, dr}{\int_0^a J_0^2\left(\frac{z_{m,0}}{a} r\right) r\, dr}$$

$$u_t(r,0) = \psi(r) = \sum_{m=1}^{+\infty} J_0\left(\frac{z_{m,0}}{a} r\right)\left(\frac{z_{m,0}}{a} c\, b_n\right)$$

$$b_n = \frac{a}{z_{m,0}\, c} \cdot \frac{\int_0^a J_0\left(\frac{z_{m,0}}{a} r\right)\psi(r)\, r\, dr}{\int_0^a J_0^2\left(\frac{z_{m,0}}{a} r\right) r\, dr}$$

4. $$u(r,\theta,t) = \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} J_n\left(\frac{z_{m,n}}{a} r\right) e^{-k\frac{z_{m,n}^2}{a^2} t}\, a_{m,n}$$

Now $\phi(r,\theta,0) = 1 - 2\cos\theta \Rightarrow$ only $n=0$, $n=1$ are allowed.

$$\Rightarrow \quad 1 = \sum_{m=1}^{+\infty} J_0\left(\frac{z_{m,0}}{a} r\right) a_{m,0} \Rightarrow a_{m,0} = \frac{\int_0^a J_0\left(\frac{z_{m,0}}{a} r\right) r\, dr}{\int_0^a J_0^2\left(\frac{z_{m,0}}{a} r\right)^2 r\, dr}$$

$$-2 = \sum_{m=1}^{+\infty} J_1\left(\frac{z_{m,1}}{a} r\right) a_{m,1} \Rightarrow a_{m,1} = \frac{-2\int_0^a J_1\left(\frac{z_{m,1}}{a} r\right) r\, dr}{\int_0^a J_1^2\left(\frac{z_{m,1}}{a} r\right)^2 r\, dr}$$

$$u(r,\theta,t) = \sum_{m=1}^{+\infty} a_{m,0} J_0\left(\frac{z_{m,0}}{a} r\right) e^{-k\frac{z_{m,0}^2}{a^2} t} + \sum_{m=1}^{+\infty} a_{m,1} J_1\left(\frac{z_{m,0}}{a} r\right) e^{-k\frac{z_{m,1}^2}{a^2} t}$$

5. Method of Separation of Variables $\Rightarrow$

Step 1

$$\frac{R''+\frac{1}{r}R'}{R}+\frac{Z''}{Z}=\frac{T'}{kT}=-\lambda.$$

$$\frac{Z''}{Z}=-\mu$$

$$\Rightarrow \quad Z''+\mu Z=0, \quad Z(0)=Z(b)=0$$

$$R''+\frac{1}{r}R'+(\lambda-\mu)R=0; \quad R'(a)=0.$$

$$T'+\lambda kT=0.$$

step 2. $\mu_n=\left(\frac{n\pi}{b}\right)^2$, $n=1,2,\cdots$, $Z(z)=\sin\left(\frac{n\pi}{b}z\right)$

Then $\lambda-\mu_n=\frac{{z'_{m,0}}^2}{a^2}$.

$R=J_0\left(\frac{z'_{m,0}}{a}r\right)$, where $J_0'(z'_{m,0})=0$,

Note that $z'_{0,0}=0 \Rightarrow m=0$, $R_0=1$.

So $m=0$, $\lambda_{0,n}=\mu_n$, $R_0=1$, $Z_n=\sin\left(\frac{n\pi}{b}z\right)$

$m\geq 1$, $\lambda_{m,n}=\mu_n+\frac{{z'_{m,0}}^2}{a^2}$, $R_m=J_0\left(\frac{z'_{m,0}}{a}r\right)$, $Z_n=\sin\left(\frac{n\pi}{b}z\right)$.

step 3. Sum-up.

$$u(r,\theta,z,t)=\sum_{n=1}^{+\infty}a_{0,n}\sin\left(\frac{n\pi}{b}z\right)e^{-k\left(\frac{n\pi}{b}\right)^2t}+\sum_{n}\sum_{m=1}^{+\infty}a_{m,n}\sin\left(\frac{n\pi}{b}z\right)J_0\left(\frac{z'_{m,0}}{a}r\right)e^{-k\left(\left(\frac{n\pi}{b}\right)^2+\frac{(z'_{m,0})^2}{a^2}\right)t}$$

where

$$a_{0,n} = \frac{\int_0^a \int_0^b \phi(r, z) \sin\left(\frac{n\pi}{b} z\right) dz\, R_0\, r\, dr}{\int_0^a \int_0^b \sin^2\left(\frac{n\pi}{b} z\right) dz\, R_0^2\, r\, dr}$$

$$a_{m,n} = \frac{\int_0^a \int_0^b \phi(r, z) \sin\left(\frac{n\pi}{b} z\right) dz\, J_0\left(\frac{z_{m,0}}{a} r\right) r\, dr}{\int_0^a \int_0^b \sin^2\left(\frac{n\pi}{b} z\right) dz\, J_0^2\left(\frac{z_{m,0}}{a} r\right) r\, dr}$$