

A Complete List of Formulas (and Theorems) in MATH400-201

Part I. First Order Equations

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

1. Method of Characteristics:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

Suppose that the initial conditions are given by $(x_0(\xi), y_0(\xi), u_0(\xi))$. Then we need to solve

$$\begin{cases} \frac{dx}{ds} = a(x, y, u), x(0) = x_0(\xi) \\ \frac{dy}{ds} = b(x, y, u), y(0) = y_0(\xi) \\ \frac{du}{ds} = c(x, y, u), u(0) = u_0(\xi) \end{cases}$$

2. General Solutions for

$$a(x, y)u_x + b(x, y)u_y = c(x, y)$$

Method: 1) Solve the characteristics:

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$$

to get $F(x, y; \lambda) = 0$ and then solve $\lambda = f(x, y)$. 2) Change variable

$$x' = x, \lambda = f(x, y), u(x, y) = U(x', \lambda)$$

New equation for U :

$$aU_{x'} = c$$

and integrate

3. Traffic Flow Problem:

$$\rho_t + c(\rho)\rho_x = 0, \rho(x, 0) = \rho_0(x)$$

where $Q'(\rho) = c(\rho)$.

General solution:

$$x - \xi = c(\phi(\xi))t$$

Shock:

$$\frac{ds}{dt} = \frac{Q(\rho_+) - Q(\rho_-)}{\rho_+ - \rho_-}, s(t_0) = x_0$$

Expanding fan:

$$u = H(\Lambda) = H\left(\frac{x}{t}\right), \text{ where } c(H) = \Lambda$$

4. Fully nonlinear problem:

$$F(x, y, u, u_x, u_y) = 0$$

Charpit's equation:

$$\begin{cases} \frac{dx}{ds} = F_p, x(0) = x_0(\xi) \\ \frac{dy}{ds} = F_q, y(0) = y_0(\xi) \\ \frac{dp}{ds} = -F_x - pF_u, p(0) = p_0(\xi) \\ \frac{dq}{ds} = -F_y - qF_u, q(0) = q_0(\xi) \\ \frac{du}{ds} = pF_p + qF_q, u(0) = u_0(\xi) \end{cases}$$

where (p_0, q_0) is computed via

$$F(x_0, y_0, p_0, q_0) = 0, u'_0 = p_0x'_0 + q_0y'_0$$

Part II. Second order PDEs: general Formula

1. Wave Equation on the whole line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t), -\infty < x < +\infty, \\ u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), -\infty < x < +\infty \end{cases} \quad (1)$$

D'Alembert's formula

$$u(x, t) = \frac{1}{2}(\phi(x - ct) + \phi(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds + \frac{1}{2c} \int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy \right) ds$$

2. Diffusion Equation on the whole line:

$$\begin{cases} u_t - k u_{xx} = f(x, t), -\infty < x < +\infty, t > 0 \\ u(x, 0) = \phi(x), -\infty < x < +\infty \end{cases} \quad (2)$$

Solution formula

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \phi(y) dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s) dy ds$$

3. Wave Equation on the half line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t), 0 < x < +\infty, t > 0 \\ u(x, 0) = \phi, u_t(x, 0) = \psi, 0 < x < +\infty \\ u(0, t) = 0 \end{cases} \quad (3)$$

Method of Reflection: extend f, ϕ, ψ oddly to $(-\infty, +\infty)$.

There is a similar formula for Neumann boundary condition.

Inhomogeneous BC: $u(0, t) = h(t)$. Use $V(x, t) = u(x, t) - xh(t)$.

4. Diffusion Equation on the half line:

$$\begin{cases} u_t - k u_{xx} = f(x, t), 0 < x < +\infty, t > 0 \\ u(x, 0) = \phi, 0 < x < +\infty \\ u(0, t) = 0 \end{cases} \quad (4)$$

Method of Extension: extend f and ϕ oddly. Solution formula:

$$u(x, t) = \int_0^{\infty} (S(x - y, t) - s(x + y, t) \phi(y)) dy + \int_0^t \int_0^{\infty} (S(x - y, t - s) - S(x + y, t - s)) f(y, s) dy ds$$

III: Boundary Value Problems and Method of Separation of Variables

1. Method of Separation of Variations

Step 1: Find the right separated functions. Plug into PDE and BC (homogeneous or natural BC). Distinguish EVP and ODE.

Step 2: Solve (EVP) and (ODE)

Step 3: Sum-up. Plug in the inhomogeneous BC.

2. Standard Eigenvalue Problems

$$X'' + \lambda X = 0, 0 < x < l$$

2.1) Dirichlet BC: $X(0) = X(l) = 0$

$$\lambda_n = \frac{(n\pi)^2}{l^2}, X_n = \sin\left(\frac{n\pi}{l}x\right), n = 1, 2, \dots$$

2.2) Neumann BC: $X'(0) = X'(l) = 0$

$$\lambda_n = \frac{(n\pi)^2}{l^2}, X_n = \cos\left(\frac{n\pi}{l}x\right), n = 0, 1, 2, \dots$$

2.3) Periodic BC: $X(0) = X(l), X'(0) = X'(l)$

$$\lambda_n = \frac{(2n\pi)^2}{l^2}, X_n = a \cos\left(\frac{2n\pi}{l}x\right) + b \sin\left(\frac{2n\pi}{l}x\right), n = 0, 1, 2, \dots$$

2.4) Summary of Robin boundary condition eigenvalue problems

$$\begin{cases} X'' + \lambda X = 0, 0 < x < l, \\ X'(0) - a_0 X(0), X'(l) + a_l X(l) = 0 \end{cases}$$

Hyperbola:

$$a_0 + a_l + a_0 a_l l = \left(a_0 + \frac{1}{l}\right)\left(a_l + \frac{1}{l}\right) - \frac{1}{l^2} = 0$$

divides the parameter space (a_0, a_l) into Five Regions. Depending on the regions, the number of negative or zero eigenvalues can be determined.

Equation for negative eigenvalues:

$$\lambda = -\gamma^2, \tanh(\gamma l) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}, X = \cosh(\gamma x) + \frac{a_0}{\gamma} \sinh(\gamma x)$$

Equation for zero eigenvalue

$$a_0 + a_l + a_0 a_l l = 0, \lambda = 0, X = 1 - a_0 x$$

Equation for positive eigenvalue

$$\lambda = \beta^2, \tan(\beta l) = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}$$

3. Sturm-Liouville Eigenvalue Problem

$$(p(x)X')' + \lambda w(x)X = 0, 0 < x < l,$$

$$X'(0) - h_0 X(0) = 0, X'(l) + h_1 X(l) = 0$$

Lagrange's identity:

$$\int_0^l [f(pg')' - g(pf')'] = (pf'g' - pf'g)|_0^l$$

1) all eigenvalues are real

2) $\lambda_1 > 0$ if $h_0 > 0, h_1 > 0$

3) Different eigenfunctions are orthogonal with respect to the weight function w :

$$\int_0^l w(x)X_n X_m dx = 0$$

4) eigenvalues are discrete and approach to infinity

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots, \lambda_n \rightarrow +\infty$$

5) Expansion with respect to the eigenfunctions:

$$f(x) = \sum_n A_n X_n(x)$$

$$A_n = \frac{\int_0^l X_n f w(x) dx}{\int_0^l X_n^2 w(x) dx}$$

4. Bessel functions:

$$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R + \lambda R = 0, 0 \leq r < a, R(a) = 0$$

Then

$$\lambda = \frac{z_{m,n}^2}{a^2}, R(r) = J_n\left(\frac{z_{m,n}}{a}r\right), m = 1, 2, \dots$$

where Bessel function of order zero of first kind:

$$J_0'' + \frac{1}{z}J_0' + J_0 = 0, J_0(0) = 1$$

Bessel function of order n :

$$J_n'' + \frac{1}{z}J_n' + J_n - \frac{n^2}{z^2}J_n = 0, J_n(z) \sim z^n$$

The zeroes of J_n is denoted as

$$0 < z_{1,n} < z_{2,n} < \dots < z_{m,n} < \dots$$

5. Method of Separation of Variables for heat equation/wave equation

1) Diffusion equation without source:

$$u_t = ku_{xx}, 0 < x < l$$

$$u(x, 0) = \phi$$

$$u(0, t) = 0, u(l, t) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t} \sin\left(\frac{n\pi}{l}x\right)$$

where

$$a_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi}{l}x\right) dx$$

Similar formula for wave equation.

2) Diffusion equation with source:

$$u_t = ku_{xx} + f(x, t),$$

$$u(x, 0) = \phi$$

$$u(0, t) = h(t), u(l, t) = k(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin\left(\frac{n\pi}{l}x\right)$$

Then we need to solve

$$u_n' + k\lambda_n u_n = \frac{2n\pi}{l^2} (h(t) - (-1)^n k(t)) + f_n(t)$$

$$u_n(0) = \phi_n$$

3) Wave equation with source:

$$u_{tt} = c^2 u_{xx} + f(x, t),$$

$$u(x, 0) = \phi, u_t(x, 0) = \psi$$

$$u(0, t) = h(t), u(l, t) = k(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin\left(\frac{n\pi}{l}x\right)$$

Then we need to solve

$$u_n'' + c^2 \lambda_n u_n = \frac{2n\pi}{l^2} (h(t) - (-1)^n k(t)) + f_n(t)$$

$$u_n(0) = \phi_n, u_n'(0) = \psi_n$$

6 Method of separation of variables applied to Laplace equation

6.1) Laplace equation in rectangle and cubes

$$u(x, y) = X(x)Y(y)$$

6.2) Radial Domains

$$u(r, \theta) = R(r)\Theta(\theta)$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}$$

1) solution to

$$\Delta u = 0, 0 \leq r < a, 0 \leq \theta < 2\pi, u(a, \phi) = h(\phi)$$

is given by

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) d\phi, a_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos(n\phi) d\phi, b_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin(n\phi) d\phi,$$

6.4) Laplace equation on wedges, annulus, exterior of disk

7. Method of separation of variables for Diffusion equation in a disk: polar coordinate

$$u_t = k(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta})$$

$$u(r, \theta, 0) = \phi(r, \theta)$$

$$u(a, \theta, t) = 0$$

The solution is given by

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} J_n\left(\frac{z_{m,n}}{a}r\right) (a_{m,n} \cos(n\theta) + b_{m,n} \sin(n\theta)) e^{-k \frac{z_{m,n}^2}{a^2} t}$$

where

$$a_{m,n} = \frac{\int_0^a \int_0^{2\pi} \phi(r, \theta) \cos(n\theta) d\theta J_n\left(\frac{z_{m,n}}{a}r\right) r dr}{\pi \int_0^a J_n^2\left(\frac{z_{m,n}}{a}r\right) r dr}$$

$$b_{m,n} = \frac{\int_0^a \int_0^{2\pi} \phi(r, \theta) \sin(n\theta) d\theta J_n\left(\frac{z_{m,n}}{a} r\right) r dr}{\pi \int_0^a J_n^2\left(\frac{z_{m,n}}{a} r\right) r dr}$$

Part IV: Properties of Laplace equation

1. Classification of second order equations

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y = 0$$

Type of PDEs: Elliptic, Parabolic, Hyperbolic

Change of variables to standard form

$$\partial_x = a_{11}\partial_\xi + a_{12}\partial_\eta$$

$$\partial_y = a_{21}\partial_\xi + a_{22}\partial_\eta$$

Then

$$\xi = a_{11}x + a_{21}y$$

$$\eta = a_{12}x + a_{22}y$$

2. The energy of wave equation

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dt + \frac{c^2}{2} \int_{-\infty}^{+\infty} u_x^2(x, t) dx$$

$$\frac{dE}{dt} = 0$$

The energy of diffusion equation

$$E(t) = \frac{1}{2} \int_{-\infty}^{+\infty} u(x, t) dx$$

$$\frac{dE}{dt} \leq 0$$

Uniqueness of wave and diffusion equations.

3. For Laplace equation

$$\Delta u = f \text{ in } D$$

3.1) Uniqueness: The solution to Dirichlet BC is unique; the solution to Neumann BC is unique, up to a constant; the solution to Robin BC is unique provided $a \geq 0, a \neq 0$.

Uniqueness proved by energy method (and divergence theorem).

3.2) Laplace equation on a disk: Poisson formula

$$\Delta u = 0 \text{ in } 0 \leq r < a, 0 \leq \theta < 2\pi, u(a, \theta) = h(\theta)$$

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 + r^2 - 2ar \cos(\theta - \phi)} d\phi$$

3.3) Mean Value Theorem: If $\Delta u = 0$ then

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(a, \phi) d\phi$$

3.4) Maximum Principle: If $\Delta u = 0$ in D then $\max_D u = \max_{\partial D} u$ and equality holds if and only if $u \equiv \text{Constant}$