Midterm Examination for MATH400-201

Total: 100 points

Answer All Questions. Show All Steps.

Date: Feb. 23rd, 2016

1. (10pts) Solve the following first order PDE:
$$xu_x + (y+x)u_y = (u+x), \quad u(x,2x) = 0, -\infty < x < +\infty$$

2. (20pts) Find the general solutions to the following first order PDE
$$u_x + (2x)u_y = yu$$

3. (30pts) Consider the traffic flow problem
$$\rho_t + (1-2\rho)\rho_x = 0, \ t > 0$$

Solve for $\rho(x,t)$ with the following initial conditions

$$\rho(x,0) = \begin{cases} 1, \text{ when } \ x < 0; \\ 0, \text{ when } \ x > 0 \end{cases}$$
$$\rho(0-,t) = 2, t > 0$$

4. (20pts) Solve the following second order PDE:
$$u_{tt} - 3u_{tx} = 0$$
$$u(x,0) = 0, u_t(x,0) = x$$

5. (20pts) Solve the following wave equation:
$$u_{tt} = c^2 u_{xx} + x, -\infty < x < +\infty, t > 0$$
$$u(x,0) = \sin x, u_t(x,0) = e^x$$

List of Formulas

(1) First Order Fully Nonlinear PDE

$$F(x,y,u,u_x,u_y)=0$$

$$\begin{cases} \frac{dx}{ds}=F_p, x(0)=x_0(\xi), \\ \frac{dy}{ds}=F_q, y(0)=y_0(\xi), \\ \frac{dp}{ds}=-F_x-pF_u,\ p(0)=p_0(\xi) \\ \frac{dq}{ds}=-F_y-qF_u,\ q(0)=q_0(\xi) \\ \frac{du}{ds}=pF_p+qF_q,\ u(0)=u_0(\xi) \end{cases}$$

$$F(x_0,y_0,u_0,p_0,q_0)=0, u_0^{'}=p_0x_0^{'}+q_0y_0^{'}$$

(2) General solutions of first order $au_x+bu_y=c$:

$$x^{'}=x,\ y^{'}=\xi=\xi(x,y)$$

$$aU_{x'}=c$$

(3) Consider quasi-linear first order PDE

$$u_t+c(u)u_x=0,\quad Q(u)=\int_0^u c(u)du$$

Equation for shock curve: $x=s(t)$

$$\frac{ds}{dt}=\frac{Q(u^+)-Q(u^-)}{u^+-u^-}$$

Equation for expansion fan: $u=U(\frac{x}{t})$

$$c(U)=\frac{x}{t}$$

(4) Change of Variables

$$\partial_t=b_{11}\partial_\xi+b_{12}\partial_\eta$$

$$\partial_x=b_{21}\partial_\xi+b_{22}\partial_\eta$$

Then

$$\xi=b_{11}t+b_{21}x$$

$$\eta=b_{12}t+b_{22}x$$

(5) d'ALembert's formula:

$$u_{tt}=c^2u_{xx}+f(x,t)$$

$$u(x,0)=\phi(x), u_t(x,0)=\psi(x)$$

$$u(x,t)=\frac{1}{2}[\phi(x-ct)+\phi(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds+\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}f(y,s)dyds$$