Solutions to Midterm Test

1. initial curve and data can be parametrized

$$x_0(\xi)=\xi, \quad y_0(\xi)=2\xi, \quad u_0(\xi)=0.$$

Hence

$$\begin{cases} \dfrac{dx}{ds}=x, \quad x(0)=\xi & (1) \\ \dfrac{dy}{ds}=y+x, \quad y(0)=2\xi & (2) \\ \dfrac{du}{ds}=u+x, \quad u(0)=0 & (3) \end{cases}$$

From (1) $\Rightarrow$ $x=\xi e^s$

Substituting into (2) : $\dfrac{dy}{ds}=y+\xi e^s, \quad y(0)=2\xi$

$$y=\xi s e^s + A e^s \Rightarrow y=\xi s e^s+2\xi e^s$$

Substituting into (3) : $\dfrac{du}{ds}=u+\xi e^s, \quad u(0)=0$

$$u=\xi s e^s + A e^s \Rightarrow u=\xi s e^s$$

$$x=\xi e^s$$
$$y=\xi s e^s+2\xi e^s$$
$$u=\xi s e^s = y-2\xi e^s = y-2x$$

#

2. Characteristics:

$$\frac{dx}{1} = \frac{dy}{2x} \Rightarrow \frac{dy}{dx} = 2x$$

$$\Rightarrow y = x^2 + \xi$$

$$\xi = y - x^2$$

Let $x' = x$
$y' = \xi = y - x^2$, $u = U$

Then $u_x + 2x u_y = U_{x'} = yU = (x^2 + \xi) U$

$$\frac{dU}{U} = (x'^2 + \xi)\, dx'$$

$$\ln U = \frac{x'^3}{3} + \xi x' + C(\xi)$$

$$U = e^{C(\xi)}\, e^{\frac{1}{3}x'^3 + \xi x'}$$

So

$$u = U = f(\xi)\, e^{\frac{1}{3}x^3 + (y - x^2)x}$$

$$= f(y - x^2)\, e^{-\frac{2}{3}x^3 + xy}$$

where f is arbitrary

#.

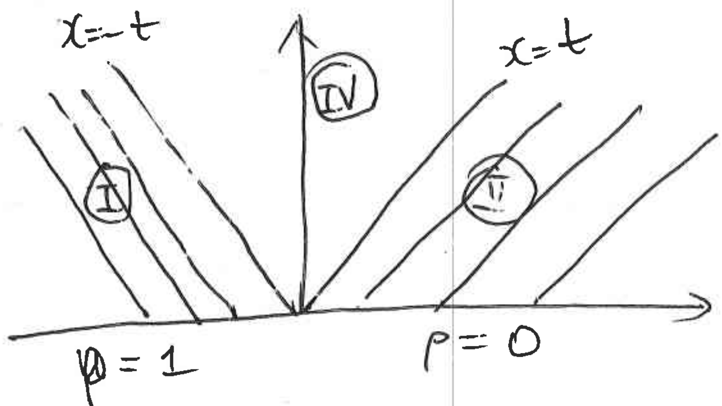
3. $c(\rho) = 1 - 2\rho$. $Q(\rho) = \rho - \rho^2$

For the first initial condition

$$x - \xi = c(\rho)\, t$$

$\xi < 0$, $x - \xi = c(1)\, t = -t \Rightarrow x + t = \xi < 0$

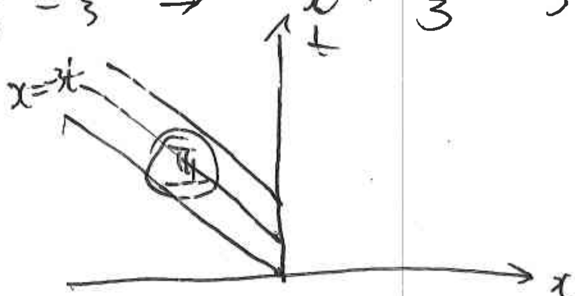
$\xi > 0$, $x - \xi = c(0)\, t = t \Rightarrow x - t = \xi > 0$



For the second initial condition

$$t - \frac{x}{c(\rho)} = \xi, \quad \xi > 0, \quad x < 0.$$

$$t - \frac{x}{c(2)} = \xi \Rightarrow t + \frac{x}{3} = \xi > 0$$



There is a shock curve between I and III.

$x = s(t)$,

$$\frac{ds}{dt} = \frac{Q(\rho^+) - Q(\rho^-)}{\rho^+ - \rho^-} = \frac{Q(2) - Q(1)}{2-1}$$

$$= \frac{2-4-(1-1^2)}{2-1} = -2$$

$s(0) = 0$

So $x = s(t) = -2t$ is the shock curve.

We need to insert an expansion fan between II and IV:

$$c(u) = \frac{x}{t} \qquad 1-2u = \frac{x}{t} \quad \Rightarrow \quad u = \frac{1}{2}\left(1 - \frac{x}{t}\right)$$

Hence the final soln is

$$u(x,t) = \begin{cases} 1, & x < -2t \\ 2, & -2t < x < 0 \\ \frac{1}{2}\left(1 - \frac{x}{t}\right), & 0 < x < t \\ 0, & x > t \end{cases}$$

4. $\partial_t^2 - 3\partial_t\partial_x$

$= \partial_t(\partial_t - 3\partial_x) = 0.$

$\partial_\xi = \partial_t$ 4 $\quad \partial_\eta = \partial_t - 3\partial_x$ 4

16/20 not attempt to find f & g 18/20 mistake 19/20 d'Alembert wrong ICs

$\Rightarrow \quad \partial_t = \partial_\xi$

$\partial_x = \frac{1}{3}\partial_\xi - \frac{1}{3}\partial_\eta \qquad \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$

$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \Rightarrow \xi = t + \frac{x}{3}$

$\eta = -\frac{x}{3}$ 4

So $u = F(\xi) + G(\eta) = F(t + \frac{x}{3}) + G(-\frac{x}{3})$

$= f(x+3t) + g(x)$ 4

$u(x,0) = 0 \Rightarrow f(x) + g(x) = 0 \quad \Rightarrow g(x) = -\frac{x^2}{6}$

$u_t(x,0) = x \Rightarrow 3f'(x) = x \quad \Rightarrow f(x) = \frac{x^2}{6}$

$u = \frac{1}{6}(x+3t)^2 - \frac{1}{6}x^2$ 4

$\Rightarrow$ 20 pt.

5. Use d'Alembert's formula

$u = \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} e^s\, ds$

$+ \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} y\, dy\, ds$

10

$$= \sin x \cos ct + \frac{1}{c} e^x \sinh ct + \frac{1}{2c} \int_0^t \frac{1}{2} y^2 \Big|_{x-c(t-s)}^{x+c(t-s)} ds$$ 5

$$= \sin x \cos ct + \frac{1}{c} e^x \sinh ct + \frac{1}{2c} \int_0^t 2xc(t-s)\, ds$$

$$= \sin x \cos ct + \frac{1}{c} e^x \sinh ct + \frac{x}{2} t^2$$ 5

10 pts for d'Alembert

5 for each integral