

A Summary of 1st order PDEs

Case 1: $a(x, y) u_x + b(x, y) u_y = 0$

- Method: ① $\frac{dx}{a} = \frac{dy}{b} \Rightarrow \zeta = F(x, y)$ (characteristic)
- ② general solution $u = f(\zeta) = f(F(x, y))$
- ③ Plug in IC

Case 2: $a(x, y) u_x + b(x, y) u_y = c(x, y, u)$

- Method 1: ① $\frac{dx}{a} = \frac{dy}{b} \Rightarrow \zeta = F(x, y)$
- ② change of variables $\begin{cases} x' = x \\ y' = F(x, y) \\ U = u \end{cases} \Rightarrow a U_{x'} = c \Rightarrow u = U$
- ③ Plug in IC

- Method 2: Parametrize IC and Initial Data $(x_0(\zeta), y_0(\zeta), u_0(\zeta))$
- $\begin{cases} \frac{dx}{ds} = a, & x(0) = x_0(\zeta) \\ \frac{dy}{ds} = b, & y(0) = y_0(\zeta) \\ \frac{du}{ds} = c, & u(0) = u_0(\zeta) \end{cases} \Rightarrow \begin{cases} x = x(s; \zeta) \\ y = y(s; \zeta) \\ u = u(s; \zeta) \end{cases} \Rightarrow u =$

Case 3: $u_t + c(u) u_x = 0, \quad u(x, 0) = f(x)$

(1) $x = (f(\zeta))t + \zeta, \quad u = f(\zeta)$

(2) breaking time $t_B = \frac{1}{\max |c'(f(\zeta)) f'(\zeta)|}$

(3) Expansion Fan: $u = H(\zeta), \quad c(H(\zeta)) = \zeta, \quad \zeta = \frac{x}{t}$

(4) Shock: $\frac{ds}{dt} = \frac{Q(u^+) - Q(u^-)}{u^+ - u^-}, \quad s(t_0) = x_0; \quad Q = \int c(u) du$

Case 4: $F(x, y, u, u_x, u_y) = 0$

- ① Parametrize IC and I Data, $(x_0(\zeta), y_0(\zeta), u_0(\zeta))$
- ② $F(x_0, y_0, u_0, p_0, q_0) = 0, \quad u_0' = p_0 x_0' + q_0 y_0' \Rightarrow p_0 =, \quad q_0 =$

③ Charpit's Equations $\begin{cases} \frac{dx}{ds} = F_p, & x(0) = x_0(\zeta), & \frac{dp}{ds} = -F_x - F_u p, & p(0) = p_0 \\ \frac{dy}{ds} = F_q, & y(0) = y_0(\zeta), & \frac{dq}{ds} = -F_y - F_u q, & q(0) = q_0 \\ \frac{du}{ds} = p F_p + q F_q, & u(0) = u_0(\zeta) \end{cases}$