Solutions to Midterm Examination

1. (i) Characteristics:

$$\frac{dx}{y} = \frac{dy}{x} \Rightarrow x^2 - y^2 = \lambda \qquad (2\text{ pts})$$

Change of variable

$$\begin{cases} \lambda = x^2 - y^2 \\ x' = x \end{cases}, \quad u(x,y) = U(\lambda, x')$$

The equation becomes

$$y u_x + x u_y = y U_{x'} = y^3 u = y^3 U$$

$$U_{x'} = y^2 U = (x^2 - \lambda) U = (x'^2 - \lambda) U \qquad (4\text{ points})$$

$$\frac{dU}{U} = (x'^2 - \lambda)\, dx'$$

$$\ln U = \int (x'^2 - \lambda)\, dx' = \frac{1}{3} x'^3 - \lambda x' + C$$

$$U = c\, e^{\frac{1}{3} x'^3 - \lambda x'}$$

c depends on λ so the general solution is

$$u(x,y) = U = f(\lambda)\, e^{\frac{1}{3} x^3 - (x^2 - y^2) x} = f(x^2 - y^2)\, e^{-\frac{2}{3} x^3 + x y^2} \qquad (4\text{ pts})$$

(ii) Substituting the general solution to the initial condition

$$1 = f(0)\, e^{-\frac{2}{3} x^3 + x^3} \Rightarrow f(0) = e^{-\frac{1}{3} x^3} \qquad (5\text{pts})$$

This can never happen.

So there is no solution for this problem (5pts)

(ii) Substituting the general solution to the initial condition

$$y = f(1-y^2)\, e^{-\frac{2}{3}+y^2}$$

$$\Rightarrow \quad y\, e^{\frac{2}{3}-y^2} = f(1-y^2) \qquad \text{—— (2pts)}$$

Let $1-y^2=\sigma \Rightarrow y^2 = 1-\sigma \Rightarrow y = \sqrt{1-\sigma}$

$2 \le y \le 3 \Rightarrow 4 \le 1-\sigma \le 9 \Rightarrow -8 \le \sigma \le -3$ (2pts)

$$f(\sigma) = \sqrt{1-\sigma}\; e^{\frac{2}{3}-(1-\sigma)} = \sqrt{1-\sigma}\; e^{-\frac{1}{3}+\frac{2}{3}\sigma}$$

2pts

for $-8 \le \sigma \le -3$

So the solution is

$$u = f(x^2-y^2)\, e^{-\frac{2}{3}x^3+xy^2}$$

$$= \sqrt{1-(x^2-y^2)}\; e^{-\frac{1}{3}+\frac{2}{3}(x^2-y^2)}\; e^{-\frac{2}{3}x^3+xy^2} \qquad \text{(2pts)}$$

where $-8 \le x^2-y^2 \le -3$. (2pts)

2. Write this equation as

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = 0, \qquad Q(\rho) = \frac{\sin(\pi\rho)}{\pi}$$

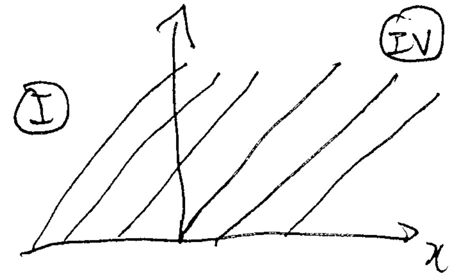
$$\frac{dt}{1} = \frac{dx}{\cos\pi\rho} = \frac{d\rho}{0}$$

(1) $\rho(x,0)=\frac{1}{4}$, $-\infty<x<+\infty$

$$\frac{dx}{dt}=\cos\pi\rho, \quad x(0)=\xi, \quad -\infty<\xi<+\infty$$

$$\frac{d\rho}{dt}=0, \quad \rho(0)=\frac{1}{4}$$

$$x=t\cos\frac{\pi}{4}+\xi=\frac{\sqrt{2}}{2}t+\xi, \quad -\infty<\xi<+\infty$$



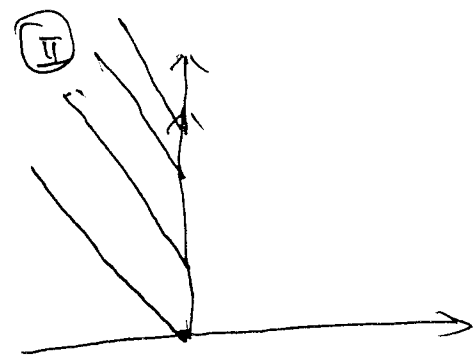
— 5pts

(2) $\rho(0-,t)=1$, $t>0$

$$\frac{dt}{dx}=\frac{1}{\cos\pi\rho}, \quad t(0)=\xi$$

$$\frac{d\rho}{dx}=0, \quad \rho(0)=1$$

$$t=\frac{1}{\cos\pi}x+\xi=-x+\xi, \quad \xi>0$$



— 5pts

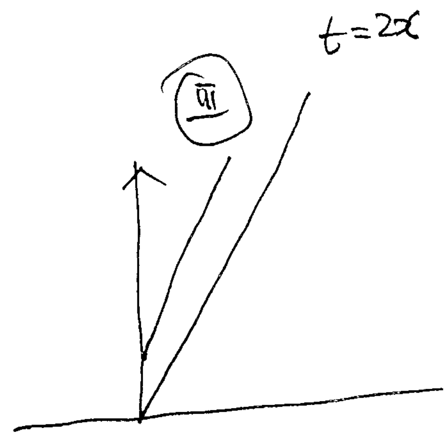
(3) $\rho(0+,t)=\frac{1}{3}$

$$\frac{dt}{dx}=\frac{1}{\cos\pi\rho}, \quad t(0)=\xi$$

$$\frac{d\rho}{dx}=0, \quad \rho(0)=\frac{1}{3}$$

$$t=\frac{1}{\cos\frac{\pi}{3}}x+\xi, \quad \xi>0$$

$$t=2x+\xi, \quad \xi>0$$

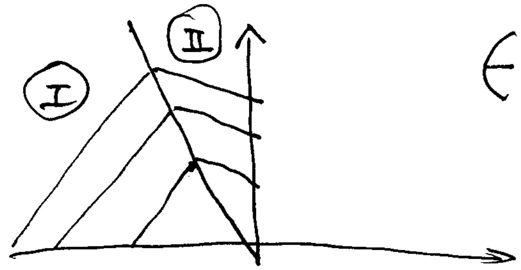


— 5pts

Between region I and II, there is a shock

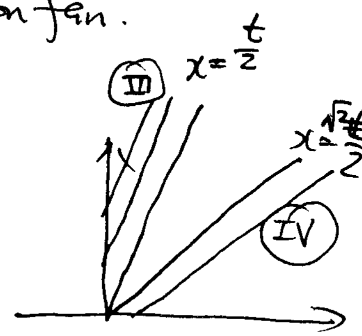
$$\begin{cases} \dfrac{ds}{dt} = \dfrac{[Q]}{[\rho]} = \dfrac{\frac{1}{\pi}\sin\frac{\pi}{4} - \frac{1}{\pi}\sin\pi}{\frac{1}{4}-1} = \dfrac{\frac{1}{\pi}\cdot\frac{\sqrt{2}}{2}}{-\frac{3}{4}} = -\dfrac{2\sqrt{2}}{3\pi} \\ s(0) = 0 \end{cases}$$

$$x = s(t) = -\frac{2\sqrt{2}}{3\pi}t$$



(−5pts)

Between region II and III, there is an expansion fan.



$$u = H\left(\frac{x}{t}\right)$$

$$c\left(H\left(\frac{x}{t}\right)\right) = \Lambda$$

$$\cos \pi H(\Lambda) = \Lambda \implies H(\Lambda) = \frac{1}{\pi}\arccos\Lambda$$

$$u = \frac{1}{\pi}\arccos\frac{x}{t}$$

(−5pts)

Finally the solution is given by

$$u = \begin{cases} \frac{1}{4}, & x < -\frac{3\sqrt{2}}{3\pi}t \\ 1, & -\frac{3\sqrt{2}}{3\pi}t < x < 0 \\ \frac{1}{3}, & 0 < x < \frac{t}{2} \\ \frac{1}{\pi}\arccos\frac{x}{t}, & \frac{t}{2} < x < \frac{\sqrt{2}t}{2} \\ \frac{1}{4} & \frac{\sqrt{2}t}{2} < x \end{cases}$$

(5pts)

Problem3: By d'Alembert's formula

$$u = \frac{1}{2}\left[\phi(x-ct)+\phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

— (3 pts)

Here $\phi(x)=x$, $\psi(x)=\sin x$, $f(x,t)=\cos ct \cos x$

(3 pts)

We compute the terms one by one:

$$\frac{1}{2}\left[\cancel{x}-ct + x+ct\right] = x$$ — (3 pts)

$$\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s\,ds = \frac{1}{2c}\left(-\cos s\right)\Big|_{x-ct}^{x+ct}$$

$$= \frac{1}{2c}\left[\cos(x-ct) - \cos(x+ct)\right] = \frac{1}{2c}\left(2\sin x \sin ct\right)$$

$$= \frac{1}{c}\sin x \sin ct$$ — (3 pts)

$$\frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos cs \cos y\,dy\,ds$$

$$= \frac{1}{2c}\int_0^t \cos(cs)\,\left(\sin y\right)\Big|_{x-c(t-s)}^{x+c(t-s)}\,ds$$

$$= \frac{1}{2c} \int_0^t \cos cs \left(\sin\left(x + c(t-s)\right) - \sin\left(x - c(t-s)\right)\right) ds$$

$$= \frac{1}{2c} \int_0^t \cos cs \cdot 2\cos x \sin c(t-s)\, ds$$

$$= \frac{1}{c}\cos x \int_0^t \left(\sin c(t-s) \cos cs\right) ds$$

$$= \frac{1}{c}\cos x \int_0^t \frac{1}{2}\left[\sin(c(t-s)+cs) - \sin(c(t-s)-cs)\right] ds$$

$$= \frac{1}{2c}\cos x \int_0^t \left[\sin ct + \sin(2cs - ct)\right] ds$$

$$= \frac{1}{2c}\cos x \left(t \sin ct + \left(-\frac{1}{2c}\cos(2cs - ct)\right)\Big|_0^t\right)$$

$$= \frac{1}{2c}\cos x \left(t \sin ct + \frac{1}{2c}(\cos ct - \cos ct)\right)$$

$$= \frac{1}{2c}\cos x\,(t \sin ct) \qquad \text{—— (6 pts)}$$

Thus the solution is given by

$$u(x,t) = x + \frac{1}{c}\sin x \sin ct$$

$$+ \frac{1}{2c}(t \sin ct)\cos x$$

— (2 pts)

Problem 4 : Since $u_x(0,t)=0$, we use even extension

$$\phi_{ext}(x) = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases} = |x|$$

So

$$u = \int_{-\infty}^{+\infty} S(x-y,t)\,|y|\,dy$$

$$= \int_0^{+\infty} \big(S(x-y,t) + S(x+y,t)\big)\, y\, dy \quad (\leftarrow 6\text{ pts})$$

Now we compute

$$\int_0^{+\infty} S(x-y,t)\, y\, dy = \frac{1}{\sqrt{4k\pi t}} \int_0^{+\infty} e^{-\frac{(x-y)^2}{4kt}}\, y\, dy$$

$$\underline{\underline{y = x+\sqrt{4kt}\,p}} \quad \frac{1}{\sqrt{4k\pi t}} \int_{-\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-p^2}(x+\sqrt{4kt}\,p)\sqrt{4kt}\,dp$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-p^2} x\, dp + \frac{\sqrt{4kt}}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4kt}}}^{+\infty} p\, e^{-p^2}\, dp$$

$$= \frac{x}{\sqrt{\pi}}\left(\int_0^{+\infty} e^{-p^2} dp + \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp\right) + \frac{\sqrt{4kt}}{\sqrt{\pi}}\left(-\frac{1}{2}e^{-p^2}\right)\Bigg|_{-\frac{x}{\sqrt{4kt}}}^{+\infty}$$

$$= \frac{x}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2} + \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp\right) + \frac{\sqrt{4kt}}{\sqrt{\pi}}\cdot\frac{1}{2}\, e^{-\frac{x^2}{4kt}}$$

$$= \frac{x}{2} + \frac{x}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp + \sqrt{\frac{kt}{\pi}}\; e^{-\frac{x^2}{4kt}}$$

— (6pts)

Similarly

$$\int_0^\infty s(x+y,t)\,y\,dy = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}}\, y\,dy$$

$$\overset{y=-x+\sqrt{4kt}\,p}{=} \frac{1}{\sqrt{4k\pi t}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-p^2}\left(-x+\sqrt{4kt}\,p\right) dp\,\sqrt{4kt}$$

$$= \frac{1}{\sqrt{\pi}}\left(-x\int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-p^2}\,dp + \sqrt{4kt}\int_{\frac{x}{\sqrt{4kt}}}^\infty p\,e^{-p^2}\,dp\right)$$

$$= +\frac{1}{\sqrt{\pi}}\left(-x\left(\int_0^\infty e^{-p^2} - \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\right) + \sqrt{4kt}\left(-\frac{1}{2}e^{-p^2}\right)\Big|_{\frac{x}{\sqrt{4kt}}}^\infty\right)$$

$$= \frac{1}{\sqrt{\pi}}\left(-\frac{\sqrt{\pi}}{2}x + x\int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp + \frac{\sqrt{4kt}}{2}\cdot e^{-\frac{x^2}{4kt}}\right)$$

— (6pts)

$u(x,t)$

$$= \frac{2}{\sqrt{\pi}}\, x \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\, dp + 2\sqrt{\frac{kt}{\pi}}\; e^{-\frac{x^2}{4kt}}$$

(2pts)