

Solutions to Final Exam

Problem (2): Method of Characteristics

$$\frac{dt}{1} = \frac{dx}{u^3} = \frac{du}{0}$$

$$\frac{dx}{dt} = u^3, \quad x(0) = \xi$$

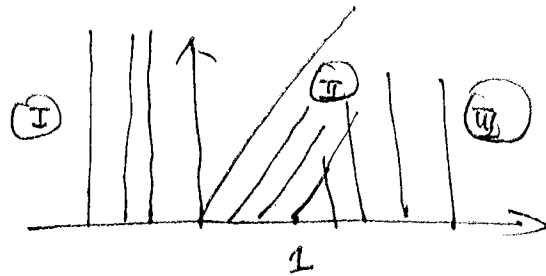
$$\frac{du}{dt} = 0, \quad u = f_0(\xi) = \begin{cases} 1, & 0 < \xi < 1 \\ 0, & \text{otherwise} \end{cases}$$

So $x = (f_0(\xi))^3 t + \xi$

For $\xi < 0$, $x = \xi$

For $0 < \xi < 1$; $x = \xi$

For $0 < \xi < 1$, $f_0(\xi) = 1$, $x = t + \xi$, $0 < \xi < 1$



5 pts

Expansion fan between (I) & (II): $u = H(\lambda)$, $\lambda = \frac{x}{t}$

$$u^3 = \lambda \Rightarrow H^3(\lambda) = \lambda \Rightarrow H = \lambda^{1/3} = \left(\frac{x}{t}\right)^{1/3}$$

3 pts

Shock between (II) and (III):

$$\begin{cases} \frac{ds}{dt} = \frac{[Q]}{[u]} \\ s(0) = 1 \end{cases} = \frac{\frac{1}{4}u_+^4 - \frac{1}{4}u_-^4}{u_+ - u_-} = \frac{\frac{1}{4} \cdot 0^4 - \frac{1}{4} \cdot 1^4}{0 - 1} = \frac{1}{4}$$

$$s = \frac{1}{4}t + 1$$

5 pts

Prbd

So the final solution is

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \left(\frac{x}{t}\right)^{\frac{1}{3}}, & 0 < x < t \\ 1, & t < x < \frac{t}{4} + 1 \\ 0, & 1 + \frac{t}{4} < x < +\infty \end{cases} \quad \text{--- (2pts)}$$

(ii). The shock occurs when the expansion fan hits the shock:

$$\begin{cases} x = t \\ x = \frac{t}{4} + 1 \end{cases} \Rightarrow t = \frac{t}{4} + 1 \Rightarrow t_B = \frac{4}{3}, \quad x_B = \frac{4}{3}$$

From $t > t_B$, we need a shock

$$\begin{cases} \frac{ds}{dt} = \frac{[Q]}{[u]} = \frac{\frac{1}{4}u_+^4 - \frac{1}{4}u_-^4}{u_+ - u_-} \\ = \frac{\frac{1}{4} \cdot 0 - \frac{1}{4} \left(\frac{8}{t}\right)^{\frac{4}{3}}}{0 - \left(\frac{8}{t}\right)^{\frac{1}{3}}} = \frac{1}{4} \left(\frac{8}{t}\right) \\ s\left(\frac{4}{3}\right) = \frac{4}{3} \end{cases}$$

$$\frac{ds}{s} = \frac{1}{4} t^{-\frac{1}{4}} dt$$

$$\ln s = \frac{1}{4} \ln t$$

$$s = C t^{\frac{1}{4}}$$

$$\begin{aligned} \frac{3}{2} s^{\frac{2}{3}} &= \frac{1}{4} \times \frac{3}{2} t^{\frac{2}{3}} + C \\ s^{\frac{2}{3}} &= \frac{1}{4} t^{\frac{2}{3}} + C \\ \left(\frac{4}{3}\right)^{\frac{2}{3}} &= \frac{1}{4} \left(\frac{4}{3}\right)^{\frac{2}{3}} + C \Rightarrow C = \frac{3}{4} \left(\frac{4}{3}\right)^{\frac{2}{3}} = \left(\frac{3}{4}\right)^{\frac{1}{3}} \\ s\left(\frac{4}{3}\right) = \frac{4}{3} &\Rightarrow \frac{4}{3} = C \left(\frac{4}{3}\right)^{\frac{1}{4}} \Rightarrow C = \left(\frac{4}{3}\right)^{\frac{3}{4}} \end{aligned}$$

So $s = \left(\frac{3}{4}\right)^{\frac{1}{3}} t^{\frac{1}{3}}$

$$s = \left(\frac{1}{4}t^{\frac{2}{3}} + \left(\frac{3}{4}\right)^{\frac{1}{3}}\right)^{\frac{3}{2}}, \quad t > \frac{4}{3} \quad \left| \begin{array}{l} 3 \text{ pts} \end{array} \right.$$

The solution when $t > t_3$ is

$$u(x, t) = \begin{cases} 0, & x < 0 \\ \left(\frac{x}{t}\right)^{\frac{1}{3}}, & 0 < x < \left(\frac{1}{4}t^{\frac{2}{3}} + \left(\frac{3}{4}\right)^{\frac{1}{3}}\right)^{\frac{3}{2}} \\ 0, & \left(\frac{1}{4}t^{\frac{2}{3}} + \left(\frac{3}{4}\right)^{\frac{1}{3}}\right)^{\frac{3}{2}} < x < +\infty \end{cases} \quad \left| \begin{array}{l} 1 \text{ pt} \end{array} \right.$$

Problem 2. (i). Consider

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & 0 < x < +\infty, t > 0 \\ u(x, 0) = 0, u_t(x, 0) = 0, & 0 < x < +\infty \\ u(0, t) = h(t), & t > 0 \end{cases}$$

The general solution of $u_{tt} - 4u_{xx} = 0$ is

$$u = f(x-ct) + g(x+ct). \quad \text{--- } 3 \text{ pts}$$

Now BC: $u(x, 0) = 0, x > 0 \Rightarrow f(x) + g(x) = 0, x > 0 \quad \text{①}$

$u_t(x, 0) = 0, x > 0 \Rightarrow -cf'(x) + g'(x) = 0, x > 0 \quad \text{②}$

$u(0, t) = h(t), t > 0 \Rightarrow f(-ct) + g(ct) = h(t), t > 0. \quad \text{3 pts}$

①, ② $\Rightarrow \begin{cases} f(x) + g(x) = 0 \\ f(x) - g(x) = A \end{cases} \Rightarrow f(x) = \frac{A}{2}, g(x) = -\frac{A}{2}, x > 0 \quad \left| \begin{array}{l} 1 \text{ pt} \end{array} \right.$

rod

$$s = ct$$

$$f(-s) + g(s) = h\left(\frac{s}{c}\right), \quad s > 0$$

$$\text{So } f(x) = \begin{cases} \frac{A}{2}, & x > 0 \\ h\left(\frac{-x}{c}\right) + \frac{A}{2}, & x < 0 \end{cases}$$

1 pts

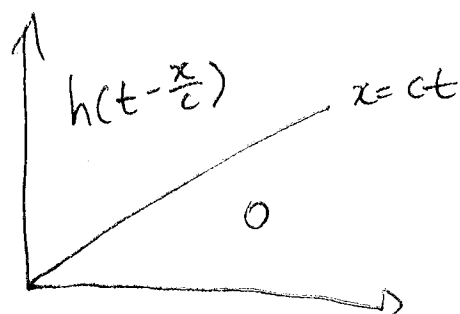
$$\text{So } u(x) = f(x-ct) + g(x+ct)$$

$$= f(x-ct) + \frac{A}{2}$$

$$= \begin{cases} 0 & x-ct > 0 \\ h\left(\frac{ct-x}{c}\right), & x-ct < 0 \end{cases}$$

2 pts

$$u(x) = \begin{cases} 0, & x > ct \\ h\left(t - \frac{x}{c}\right), & x < ct \end{cases}$$



(ii) We decompose this problem into two problems

$$u_1 \begin{cases} u_{tt} - 4u_{xx} = f(x,t), & 0 < x < t < \infty, t > 0 \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), & 0 < x < t < \infty \\ u(0,t) = 0, & t > 0 \end{cases}$$

$$u_2 \begin{cases} u_{tt} - 4u_{xx} = \textcircled{f(x,t)} \\ u(x,0) = 0, u_t(x,0) = 0 \\ u(0,t) = h(t) \end{cases}$$

3 pts

$$\text{By (ii), } u_2(x,t) = \begin{cases} 0, & x \geq ct \\ h(t - \frac{x}{c}), & x < ct \end{cases}$$

$$= \begin{cases} 0, & x > ct \\ e^{t - \frac{x}{c}}, & x < ct \end{cases}$$

Now we solve u_1 by the method of ~~separation of variables~~ ^{reflect}

$$f_{\text{ext}}(x,t) = \begin{cases} f(x,t), & x > 0 \\ f(-x,t), & x < 0 \end{cases} = xt$$

$$\phi_{\text{ext}}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$$\psi_{\text{ext}}(x) = \begin{cases} \psi(x), & x > 0 \\ -\psi(x), & x < 0 \end{cases} = \sin x$$

2 pts

2d

By d'Alembert's Formula:

$$u(x, t) = \frac{1}{2} [\phi_{\text{ext}}(x-ct) + \phi_{\text{ext}}(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{ext}}(y) dy$$

$$+ \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{\text{ext}}(y) dy ds$$

$$= \frac{1}{2} \left[\begin{cases} 1, & x > ct \\ -1, & x < ct \end{cases} + 1 \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \sin y dy$$

$$+ \frac{1}{2c} \int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} y dy \right) ds$$

$$= \frac{1}{2} \left[\begin{cases} 2, & x > ct \\ 0, & x < ct \end{cases} \right] + \frac{1}{2c} \cdot (\cos(x-ct) - \cos(x+ct))$$

$$+ \frac{1}{2c} \int_0^t \left[\frac{1}{2} [(x+c(t-s))^2 - (x-c(t-s))^2] \right] ds$$

$$= \frac{1}{2} \left[\begin{cases} 2, & x > ct \\ 0, & x < ct \end{cases} \right] + \frac{1}{c} \sin x \sin ct$$

$$+ \frac{1}{2c} \int_0^t x c(t-s) ds$$

$$\frac{1}{2} x \cdot \frac{t^2}{2}$$



4 pts

Thus

$$u(x,t) = \begin{cases} 1 + \frac{1}{2} \sin x \sin ct + \frac{1}{2} xt^2, & x > ct \\ \left(1 - \frac{x}{c}\right) + \frac{1}{2} \sin x \sin ct + \frac{1}{2} xt^2, & x < ct \end{cases}$$

1 pt

(iii) Let $v = u_1 - u_2$, where u_1, u_2 are two solutions.

$$\text{Then } v_{tt} - 4v_{xx} = 0$$

$$v(x,0) = 0, \quad v_t(x,0) = 0 \quad \Rightarrow \cdot$$

$$v(0,t) = 0, \quad t > 0 \quad \Rightarrow v_t(0,t) = 0$$

1 pt

Define energy functional:

$$E(t) = \frac{1}{2} \int_0^{+\infty} v_t^2 + \frac{c^2}{2} \int_0^{+\infty} v_x^2$$

2 pts

$$\text{Then } \frac{dE}{dt} = \int v_t v_{tt} + c^2 v_x v_{tx}$$

$$= c^2 \int v_t v_{xx} + v_x v_{tx}$$

$$= c^2 \int_0^{+\infty} (v_t v_x)_x = + c^2 v_t v_x \Big|_0^{+\infty}$$

2 pts

$$= 0 - c^2 v_t(0,t) v_x(0,t) = 0.$$

2nd

Now

$$E(t) = E(0) = \frac{1}{2} \int v_t(x, t)^2 + \frac{c^2}{2} \int v_x^2(x, 0)$$

$$= 0$$

$$\text{So } E(t) = 0 \Rightarrow \int v_t^2 + c^2 \int v_x^2 = 0$$

$$\Rightarrow v_t = v_x = 0 \Rightarrow v = \text{constant}$$

$$\text{But } v(0, t) = 0 \Rightarrow v = 0.$$

1 pt

Problem 3

$$(i). \text{ step 1 } u = X(x)T(t)$$

$$X'' + 2X' + X + \lambda X = 0, \quad 0 < x < 1$$

$$T' + \lambda T = 0.$$

$$X(0) = 0, \quad 2X'(1) - X(1) = 0$$

2 pts

step 2. solve EVP

$$X'' + 2X' + X + \lambda X = 0$$

$$(e^{2x} X')' + e^{2x} X + \lambda e^{2x} X = 0.$$

3 pts

λ is real

$$\text{Case 1 } \lambda = -\gamma^2 < 0, \quad \gamma > 0.$$

$$X = e^{\gamma x}$$

$$\gamma^2 + 2\gamma + 1 + \lambda = 0$$

$$(\gamma + 1)^2 = \gamma^2$$

$$r_1 = -1 + \gamma, \quad r_2 = -1 - \gamma$$

$$\text{So } X(x) = e^{-x} (c_1 \cosh \gamma x + c_2 \sinh \gamma x)$$

$$X(0) = 0 \Rightarrow c_1 = 0$$

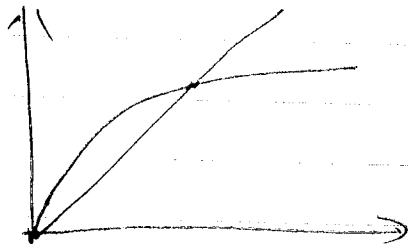
$$\text{So } X(x) = c_2 e^{-x} \sinh \gamma x$$

$$X'(x) = -c_2 e^{-x} \sinh \gamma x + \gamma c_2 e^{-x} \cosh \gamma x$$

$$2X'(1) - X(1) = 0$$

$$\Rightarrow 2(-c_2 e^{-1} \sinh \gamma + \gamma c_2 e^{-1} \cosh \gamma) - c_2 e^{-1} \sinh \gamma = 0$$

$$\tanh \gamma = \frac{2}{3} \gamma$$



$$\text{So } \exists \text{ solution } \gamma_0 \Rightarrow X_0 = e^{-x} \sinh \gamma_0 x$$

$$\text{Case 2 } \lambda = 0 \quad T = e^{\gamma_0^2 t}$$

$$r^2 + 2r + 1 = 0 \Rightarrow r = -1$$

$$X(x) = e^{-x} (c_1 + c_2 x) \quad X(0) = 0 \Rightarrow c_1 = 0$$

$$2X'(1) - X(1) = 0 \Rightarrow \text{impossible}$$

5 pts

2 pt

Case 3. $\lambda = \beta^2$, $\beta > 0$

$$(r+1)^2 + \beta^2 = 0$$

$$r = -1 \pm \beta i$$

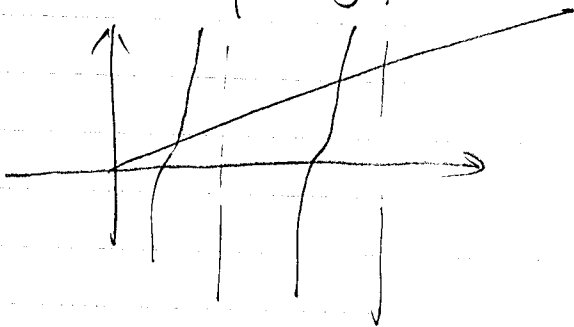
$$X(x) = e^{-x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

$$X(0) = 0 \Rightarrow c_1 = 0 \quad X(x) = e^{-x} \sin \beta x$$

$$X' = -e^{-x} \sin \beta x + \beta e^{-x} \cos \beta x$$

$$-2 \sin \beta + 2 \beta \cos \beta - \sin \beta = 0$$

$$\tan \beta = \frac{2}{3} \beta$$



$$\beta_n \in ((n-1)\pi, n\pi), \quad n=1, 2, \dots$$

$$X_n = e^{-x} \sin \beta_n x$$

$$T = e^{-\beta_n^2 t}$$

step 3: Sum up

$$U(x,t) = a_0 e^{\gamma_0^2 t} x_0(x) + \sum_{n=1}^{+\infty} a_n e^{-\beta_n^2 t} X_n(x) \quad \rightarrow 1 \text{ pt}$$

5 pts

$$\phi(x) = a_0 x_0(x) + \sum_{n=1}^{+\infty} a_n x_n(x).$$

Now the weight function is $e^{2x} = w(x)$

So

$$a_n = \frac{\int \phi x_n w}{\int x_n^2 w} \quad \text{--- 2pts.}$$

$$= \frac{\int_0^1 \phi x_n e^{2x} dx}{\int_0^1 x_n^2 e^{2x} dx} \quad n=0, 1, 2, \dots$$

(ii). As $t \rightarrow +\infty$, $e^{-\beta_n^2 t} \rightarrow 0$.

So the part $\sum_{n=1}^{+\infty} a_n e^{-\beta_n^2 t} x_n(x) \rightarrow 0$ | 2pts

So u is bounded if and only if

$$a_0 = 0$$

$$\Leftrightarrow \int_0^1 \phi(x) x_0(x) e^{2x} dx = 0$$

$$\Leftrightarrow \int_0^1 \phi(x) e^{+2x} \sinh \delta_0 x dx = 0.$$

3pts

Problem 4: u satisfies

$$\Delta u = 0 \text{ in } D$$

$$u = 1 + x^2 + 3xy.$$

$$\begin{aligned} \text{So } u(r, \theta) &= 1 + a^2 \cos^2 \theta + 3a^2 \cos \theta \sin \theta \\ &= 1 + a^2 \frac{1 + \cos 2\theta}{2} + \frac{3a^2}{2} \sin 2\theta \\ &= 1 + 2a^2 + \frac{a^2}{2} (\cos 2\theta + 3 \sin 2\theta) \end{aligned}$$

4 pts

$$\text{So } \max_{\partial D} u(a, \theta) = 1 + 2a^2 + \frac{\sqrt{1+3^2}}{2} a^2$$

$$\min_{\partial D} u(a, \theta) = 1 + 2a^2 - \frac{\sqrt{10}}{2} a^2$$

By mean-value-property

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(a, \theta) d\theta = 1 + 2a^2$$

2 pts

By Maximum Principle

$$\max_D u = \max_{\partial D} u = 1 + 2a^2 + \frac{\sqrt{10}}{2} a^2$$

2 pts

$$\min_D u = \min_{\partial D} u = 1 + 2a^2 - \frac{\sqrt{10}}{2} a^2$$

2 pts

Problem 5: First we write it in polar coordinate

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad r > 2, \quad 0 < \theta < \frac{\pi}{2}$$

$$u_{\theta}(r, 0) = 0, \quad u(r, \frac{\pi}{2}) = 0.$$

$$u(2, \theta) = a^2 \sin^2 \theta = 4 \sin^2 \theta = 4 \cdot \frac{1 - \cos 2\theta}{2} = 2(1 - \cos 2\theta)$$

u bdd.

5 pts

Method of Separation of Variables

Step 1: $u = R(r) \Theta(\theta)$

$$R'' + \frac{1}{r} R' - \frac{\lambda}{r^2} R = 0, \quad R \text{ bdd}$$

$$\Theta'' + \lambda \Theta = 0$$

$$\Theta'(0) = 0, \quad \Theta(\frac{\pi}{2}) = 0.$$

2 pts

Step 2 Solve EVP

$$\lambda = \left(\frac{(n - \frac{1}{2})\pi}{\frac{\pi}{2}} \right)^2 = (2n - 1)^2, \quad n = 1, 2, \dots$$

$$\Theta_n = \cos((2n - 1)\theta)$$

$$R = c_1 r^{2n-1} + c_2 r^{-(2n-1)} = c_2 r^{-(2n-1)}, \quad \text{bdd.}$$

5 pts

Step 3

$$u = \sum_{n=1}^{\infty} a_n r^{-(2n-1)} \cos((2n-1)\theta)$$

-3 pt

$$u(z, \theta) = \sum_{n=1}^{+\infty} a_n z^{-(2n-1)} \cos((2n-1)\theta)$$

$$= 2(1 - \cos 2\theta)$$

$$\frac{-(2n-1)}{2} a_n = \frac{\int_0^{\frac{\pi}{2}} 2(1 - \cos 2\theta) \cos((2n-1)\theta) d\theta}{\int_0^{\frac{\pi}{2}} \cos^2((2n-1)\theta) d\theta}$$

$$= \frac{8}{\pi} \cdot \int_0^{\frac{\pi}{2}} \left[\cos((2n+1)\theta) - \cos 2\theta \cos((2n-1)\theta) \right]$$

$$= \frac{8}{\pi} \left\{ \frac{1}{2n-1} \sin((2n-1) \cdot \frac{\pi}{2}) - \frac{\sin((2n+1) \frac{\pi}{2})}{2(2n+1)} \Rightarrow \frac{\sin((2n-3) \frac{\pi}{2})}{2(2n-3)} \right\}$$

5 pts

$$= \frac{8}{\pi} \left\{ \frac{(-1)^{n-1}}{2n-1} \Rightarrow \frac{(-1)^n}{2(2n+1)} \Rightarrow \frac{(-1)^n}{2(2n-3)} \right\}$$

$$= \frac{8}{\pi} (-1)^{n-1} \left\{ \frac{1}{2n-1} + \frac{1}{2(2n+1)} \Rightarrow \frac{1}{2(2n-3)} \right\}$$

$$= \frac{8}{\pi} (-1)^{n-1} \left\{ \frac{1}{2n-1} - \frac{1}{2n+1} \right\}$$

$$= \frac{16}{\pi} (-1)^{n-1} \cdot \frac{1}{4n^2-1}$$

So $u(r, \theta) = \sum_{n=1}^{+\infty} \frac{16}{\pi} (-1)^{n-1} \frac{1}{4n^2-1} \left(\frac{z}{r}\right)^{2n-1} \cos((2n-1)\theta)$