List of Formulas (and Theorems) in Math400

I. First Order Equations

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

1. Method of Characteristics:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

Initial Condition: $u = u_0(s)$ on $(x_0(s), y_0(s))$. Then

$$\frac{dx}{ds} = a, x(0) = x_0(s); \frac{dy}{ds} = b, y(0) = y_0(s); \frac{du}{ds} = c, u(0) = u_0(s)$$

Initial Condition: $u = u_0(x)$ on $(x, T, 0)$

$$\frac{dy}{dx} = \frac{b}{a}, y(0) = T; \frac{du}{dx} = \frac{c}{a}, u(0) = u_0(x)$$

Initial Condition: $u = u_0(y)$ on $(T, y, 0)$

$$\frac{dx}{dy} = \frac{a}{b}, x(0) = T; \frac{du}{dy} = \frac{c}{b}, u(0) = u_0(y)$$

2. General Solutions for

$$a(x,y)u_x + b(x,y)u_y = c(x,y)$$

Method: 1) Solve the characteristics:

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

to get $F(x,y;\lambda) = 0$ and then solve $\lambda = f(x,y)$. 2) Change variable

$$\lambda = f(x,y), x^{'} = x, u(x,y) = U(x^{'}, \lambda)$$

New equation for U:

$$aU_{x'} = c$$

and integrate

3. Traffic Flow Problem:

$$\rho_t + c(\rho)\rho_x = 0, \rho(x,0) = \rho_0(x)$$

where $Q^{'}(\rho) = c(\rho)$.

General solution:

$$x - \xi = c(\phi(\xi))t$$

Shock:

$$\frac{ds}{dt} = \frac{Q(\rho_+) - Q(\rho_-)}{\rho_+ - \rho_-}, s(t_0) = x_0$$

Expanding fan:

$$u = H(\frac{x}{t}), \text{ where } c(H(\lambda)) = \lambda$$

II. Second order equations

1. Classification of second order equations

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1 u_x + b_2 u_y = 0$$

Determinant $b^2 - 4ac$

Change of variables to standard form

2. Wave Equation on the whole line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), -\infty < x < +\infty, \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x), -\infty < x < +\infty \end{cases} \quad (1)$$

D'Alembert's formula

$$u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t (\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy)ds$$

The energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dt + \frac{c^2}{2}\int_{-\infty}^{+\infty} u_x^2(x,t)dx$$

If $f = 0$, then

$$\frac{dE}{dt} = 0$$

3. Diffusion Equation on the whole line:

$$\begin{cases} u_t - ku_{xx} = f(x,t), -\infty < x < +\infty, t > 0 \\ u(x,0) = \phi(x), -\infty < x < +\infty \end{cases} \tag{2}$$

Solution formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)f(y,s)dyds$$

4. Wave Equation on the half line:

$$\begin{cases} u_{tt} - c^2u_{xx} = f(x,t), 0 < x < +\infty, t > 0 \\ u(x,0) = \phi, \ u_t(x,0) = \psi(x), 0 < x < +\infty \\ u(0,t) = 0 \end{cases} \tag{3}$$

Method of Reflection: extend f, ϕ, ψ oddly to $(-\infty, +\infty)$.
There is a similar formula for Neumann boundary condition.
Inhomogeneous BC: $u(0,t) = h(t)$. Use $V(x,t) = u(x,t) - xh(t)$.
5. Diffusion Equation on the half line:

$$\begin{cases} u_t - ku_{xx} = f(x,t), 0 < x < +\infty, t > 0 \\ u(x,0) = \phi, 0 < x < +\infty \\ u(0,t) = 0 \end{cases} \tag{4}$$

Method of Extension: extend f and ϕ oddly. Solution formula:
$u(x,t) = \int_0^\infty (S(x-y,t) - s(x+y,t)\phi(y))dy + \int_0^t \int_0^\infty (S(x-y,t-s) - S(x+y,t-s))f(y,s)dyds$

6. Laplace equation

$$\Delta u = f \text{ in } D$$

1) Uniqueness: The solution to Dirichlet BC is unique; the solution to Neumann BC is unique, up to a constant; the solution to Robin BC is unique provided $a \geq 0, a \not\equiv 0$.
2) Method of Solving Laplace Equation: Method of Separation of Variations

Step 1: Find the right separated functions. Plug into PDE and BC (homogeneous or natural BC). Distinguish EVP and ODE.

Step 2: Solve (EVP) and (ODE)

Step 3: Sum-up.Plug in the inhomogeneous BC.

3) Laplace equation in rectangle and cubes

7. Sturm-Liouville Eigenvalue Problem

$$(p(x)X)^{'} + \lambda w(x)X = 0, 0 < x < l,$$

$$X^{'}(0) - h_0 X(0) = 0, X^{'}(l) + h_1 X(l) = 0$$

Green's identity:

$$\int_0^l [f(pg^{'})^{'} - g(pf^{'})^{'}] = (pfg^{'} - pf^{'}g)|_0^l$$

1) all eigenvalues are real
2) all eigenfunctions are simple
3) $\lambda_1 > 0$ if $h_0 > 0, h_1 > 0$
4) Different eigenfunctions are orthoonal with respect to the weight function w:

$$\int_0^l w(x)X_n X_m dx = 0$$

5) eigenvalues are discrete and approach to infinity

$$\lambda_1 < \lambda_2 < ... < \lambda_n < ..., \lambda_n \to +\infty$$

8. Method of Separation of Variables for heat equation/wave equation
1) Diffusion equation with source:

$$u_t = ku_{xx} + f(x,t),$$

$$u(x,0) = \phi$$

$$u(0,t) = h(t), u(l,t) = k(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi}{l}x)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(\frac{n\pi}{l}x)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\frac{n\pi}{l}x)$$

Then we need to solve

$$u_n^{'} + k\lambda_n u_n = \frac{2n\pi}{l^2}(h(t) - (-1)^n k(t)) + f_n(t)$$

$$u_n(0) = \phi_n$$

2) Wave equation with source:

$$u_{tt} = c^2 u_{xx} + f(x,t),$$

$$u(x,0) = \phi, u_t(x,0) = \psi$$

$$u(0,t) = h(t), u(l,t) = k(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi}{l} x)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(\frac{n\pi}{l} x)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\frac{n\pi}{l} x)$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin(\frac{n\pi}{l} x)$$

Then we need to solve

$$u_n^{''} + c^2 \lambda_n u_n = \frac{2n\pi}{l^2}(h(t) - (-1)^n k(t)) + f_n(t)$$

$$u_n(0) = \phi_n, u_n^{'}(0) = \psi_n$$

9. Method of Separation of Variables in the polar coordinate

1) solution to

$$\Delta u = 0, 0 \le r < a, 0 \le \theta < 2\pi, \ u(a,\phi) = h(\phi)$$

is given by

$$u(r,\theta) = a_0 + \sum_{n=1}^{n} r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) d\phi, a_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi \cos(n\phi) d\phi, b_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi \sin(n\phi) d\phi,$$

or Poisson's formula

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 + r^2 - 2ar\cos(\theta - \phi)} d\phi$$

2) Consequences of Poisson formula

2.1) Mean Value Theorem: If $\Delta u = 0$ then

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(a,\phi)d\phi$$

2.2) Maximum Principle: If $\Delta u = 0$ in D then $\max_{\bar{D}} u = \max_{\partial D} u$ and equality holds if and only if $u \equiv Constant$

3) Laplace equation on wedges, annulus, exterior of disk

4) Diffusion equation in polar coordinate

$$u_t = k(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta})$$

Bessel function of order zero:

$$J_0'' + \frac{1}{r}J_0 + J_0 = 0$$

Bessel function of order n:

$$J_n'' + \frac{1}{r}J_n + J_n - \frac{n^2}{r^2}J_n = 0$$

Bessel function of order ν:

$$J_n'' + \frac{1}{r}J_n + J_n - \frac{\nu^2}{r^2}J_n = 0$$