Solutions to Homework Assignment 1

1. (10pts) Solve the following first order PDE and find where the solution is defined in the $x - y$ plane.

$$u_x + xyu_y = 0, u(x, 1) = x^2$$

Solution: By the method of characteristics

$$\frac{dx}{1} = \frac{dy}{xy} = \frac{du}{0}$$

Since $u(x, 1) = x^2$, we use y as parameter so we have

$$\frac{dx}{dy} = \frac{1}{xy}, x(1) = \xi; \frac{du}{dy} = 0, u(1) = \xi^2$$

Solving the first ODE we get

$$x^2 = 2\log y + C, x^2 = 2\log y + \xi^2$$

and so $\xi^2 = x^2 - 2log(y)$

The solution is

$$u(x, y) = x^2 - 2\log(y)$$

where $y > 0$ (since the initial data curve passes through $y = 1$).

2. (20pts) Solve $xu_x + xyu_y = u$ for $u = u(x, y)$ with date $u(1, y) = y^2$ for $0 \leq y \leq 1$ and find where the solution is defined in the $x - y$ plane.

Solution: By the method of characteristics

$$\frac{dx}{x} = \frac{dy}{xy} = \frac{du}{u}$$

Since $u(1, y) = y^2$, we use x as parameter so we have

$$\frac{dy}{dx} = \frac{xy}{x}, y(1) = \xi, 0 \leq \xi \leq 1; \frac{du}{dy} = u, u(1) = \xi^2$$

Solving the first ODE we get

$$y = Ce^x, y = \xi e^{x-1}$$

and so $\xi = ye^{1-x}$

The solution is

$$u(x, y) = (ye^{1-x})^2$$

where $0 \le \xi = ye^{1-x} \le 1$.

The domain of definition is $0 \le y \le e^{x-1}$.

3. (15pts) Solve the following first order PDE and find where the solution becomes unbounded in the $x - y$ plane.

$$x^2 u_x + xyu_y = u^3, \ u = 1 \ \text{ on the curve } y = x^2$$

Solution: By the method of characteristics

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{du}{u^3}$$

Since $u = 1$ on $y = x^2$, we use s as parameter so we have

$$\frac{dx}{ds} = x^2, x(0) = \xi,$$

$$\frac{dy}{ds} = xy, y(0) = \xi^2,$$

$$\frac{du}{ds} = u^3, u(0) = 1$$

Solving the first ODE we get

$$-\frac{1}{x} = s + C, x = \frac{\xi}{1 - s\xi}$$

Solving the second ODE we get

$$\log y = -\log(1 - s\xi) + C, y = \frac{\xi^2}{1 - s\xi}$$

Solving the last ODE we get

$$-\frac{1}{2u^2} = s + C, u^2 = \frac{1}{1 - 2s}$$

Eliminating ξ and s we obtain $\xi = \frac{y}{x}, s = \frac{x^2-y}{xy}$ and so

$$u^2 = \frac{xy}{xy - 2(x^2 - y)}$$

Since $u = 1$ on $y = x^2$, we take

$$u = \sqrt{\frac{xy}{xy - 2(x^2 - y)}}$$

The blow up curve is $xy - 2(x^2 - y) = 0$.

The domain of definition is $\frac{xy}{xy-2(x^2-y)} > 0$.

4. (20pts) Solve $u_t + t^2 u_x = 4u$ for $x > 0, t > 0$ with $u(0,t) = h(t)$ and $u(x,0) = 1$.
Solution: By the method of characteristics

$$\frac{dt}{1} = \frac{dx}{t^2} = \frac{du}{4u}$$

There are two initial date curves so this problem can be decomposed into two systems of ODEs.

ODE1: For $u(x,0) = 1, x > 0$, we use t as parameter

$$\frac{dx}{dt} = t^2, x(0) = \xi, \xi > 0; \frac{du}{dt} = 4u, u(0) = 1$$

Solving the first ODE we get

$$x = \frac{1}{3}t^3 + \xi$$

Solving the second ODE we get

$$u = e^{4t}$$

in the region $\xi = x - \frac{1}{3}t^3 > 0$.

ODE2: For $u(0,t) = h(t), t > 0$, we use x as parameter

$$\frac{dt}{dx} = \frac{1}{t^2}, t(0) = \xi, \xi > 0; \frac{du}{dx} = \frac{4u}{t^2}, u(0) = h(\xi)$$

Solving the first ODE we get

$$t = (3x + \xi)^{1/3}$$

Solving the second ODE we get

$$\log u = 4(3x + \xi)^{1/3} + C, u = h(\xi)e^{-4\xi^{1/3}}e^{4(3x+\xi)^{1/3}}$$

and hence

$$u = h(t^3 - 3x)e^{-4(t^3-3x)^{1/3}}e^{4t}$$

in the region $\xi = t^3 - 3x > 0$.

In conclusion, we have

$$u(x,t) = \begin{cases} e^{4t}, \text{ if } x > \frac{t^3}{3} \\ u = h(t^3 - 3x)e^{-4(t^3-3x)^{1/3}}e^{4t}, \text{ if } x < \frac{t^3}{3} \end{cases}$$

5. (15pts) Solve $xu_x + yu_y = 2$ with date $u(x,1) = x^2$ for $-\infty < x < +\infty$. Explain why we can not determine $u(x,y)$ uniquely for $y \leq 0$.

Solution: By the method of characteristics

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{2}$$

For $u(x,1) = x^2$, we use y as parameter

$$\frac{dx}{dy} = \frac{x}{y}, x(1) = \xi; \frac{du}{dy} = \frac{2}{y}, u(1) = \xi^2$$

Solving the first ODE we get

$$x = \xi y$$

Solving the second ODE we get

$$u = 2\log y + \xi^2$$

So the solution is

$$u = 2\log y + \frac{x^2}{y^2}$$

The domain of definition is $y > 0$, since $log|y|$ is not defined at $y = 0$ and the initial data curve passes through $y > 0$.

6.(20pts) Let $u(x,y)$ solve the first order PDE

$$xu_x + yu_y = xu$$

(a). Find the general solutions. (b) Suppose we put $u = h(x)$ on $y = x$. Derive the condition that $h(x)$ must satisfy for a solution to exist.

Solution: By the method of characteristics

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xu}$$

The characteristics are $\lambda = \frac{x}{y}$.

(a). Changing variables

$$x^{'} = x, \lambda = \frac{x}{y}, u(x, y) = U(x^{'}, \lambda)$$

we have

$$xU_{x'} = x^{'}U, U_{x'} = U$$

and so

$$U = Ce^{x^{'}} = f(\lambda)e^{x^{'}}$$

The general solution is

$$u = f(\frac{x}{y})e^{x}$$

(b). When $y = x$, we have

$$u = f(1)e^{x} = h(x)$$

and so $h(x)$ must be of the form

$$h(x) = Be^{x}$$

where B is a constant.

If $h(x) = Be^{x}$ then the solution u is

$$u = f(\frac{x}{y})e^{x}, f(1) = B$$