Solutions to Homework #2

Problem 1 (20 pts): There four regions to consider.

- By the method of characteristics

$$\frac{dx}{dt} = u, \quad x(0) = \xi$$

$$\frac{du}{dt} = 0, \quad u(0) = u_0(\xi) = \begin{cases} 0, & \xi \le 0 \\ \xi, & 0 \le \xi \le 1 \\ 2-\xi, & 1 \le \xi \le 2 \\ 0, & \xi > 2 \end{cases}$$

so $\frac{dx}{dt} = u_0(\xi)$, $x(0) = \xi$ $\Rightarrow$

$$x = u_0(\xi)t + \xi, \quad u = u_0(\xi)$$

① $\xi \le 0$, $\quad u = u_0(\xi) = 0$, $\quad x = \xi$

② $0 \le \xi \le 1$, $\quad u = u_0(\xi) = \xi$, $\quad x = \xi t + \xi = \xi(t+1)$

$\Rightarrow \xi = \frac{x}{t+1}$, $\quad u = \xi = \frac{x}{t+1}$

where $0 \le \xi = \frac{x}{t+1} \le 1$

③ $1 \le \xi \le 2$, $\quad u = u_0(\xi) = 2 - \xi$, $\quad x = (2-\xi)t + \xi$

$\Rightarrow \xi = \frac{2t - x}{t-1}$, $\quad u = 2 - \xi = 2 - \frac{2t-x}{t-1} = \frac{x-2}{t-1}$

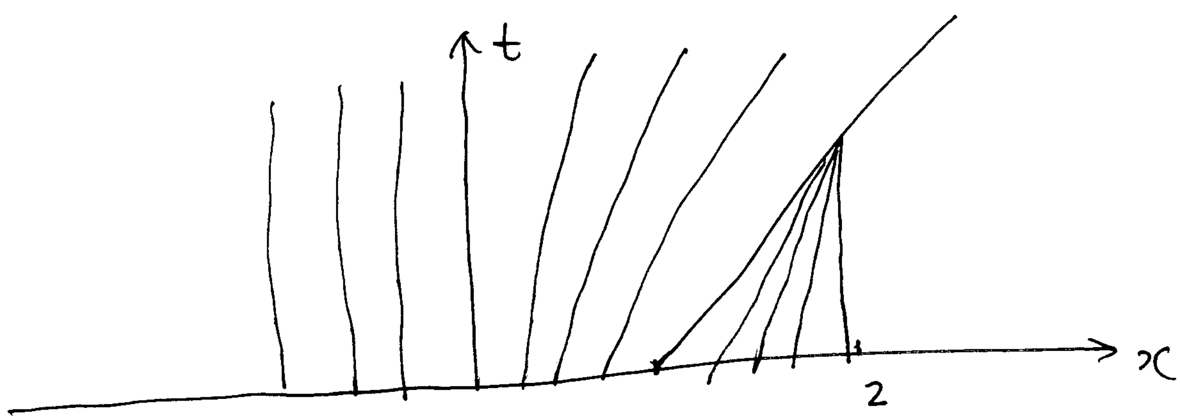
where $1 \le \xi = \frac{2t-x}{t-1} \le 2$

This solution ceases to exist when $t=1$, $x=2$

④ $\xi \geq 2$, $u = u_0(\xi) = 0$, $x = \xi$

Thus for $t < 1$, the solution u is given by

$$u(x,t) = \begin{cases} 0, & \text{if } x \leq 0,\ t<1 \\ \frac{x}{t+1}, & \text{if } 0 \leq x \leq t+1,\ t<1 \\ \frac{x-2}{t-1}, & \text{if } 1 \leq \frac{x-2t}{1-t} \leq 2,\ t<1 \iff t+1 \leq x \leq 2,\ t<1 \\ 0, & \text{if } x \geq 2,\ t<1 \end{cases}$$



The shock occurs when

$$\begin{cases} x = t+1 \\ x = (2-\xi)t + \xi, \quad 1 \leq \xi \leq 2 \end{cases}$$

has a sol'n

$$\iff t+1 = (2-\xi)t + \xi \implies (\xi - 1)(t-1) = 0 \implies t=1,\ x=2$$

So the shock occurs when $t=1$, $x=2$

Problem 2. Write the eqn as

$$\rho_t + c(\rho)\rho_x = 0, \quad -\infty < x < \infty, \ t > 0$$

where $c(\rho) = Q'(\rho) = U_{max}\left(1 - \frac{2\rho}{\rho_j}\right)$

So by the method of characteristics

$$\frac{dx}{dt} = c(\rho), \quad x(0) = \xi$$

$$\frac{d\rho}{dt} = 0, \quad \rho = \rho_0(\xi)$$

$$\Rightarrow \quad x = c(\rho_0(\xi))t + \xi, \quad \rho = \rho_0(\xi)$$

(a) (10 pts) $\rho_0(\xi) = 2 - \xi, \quad -\infty < \xi < +\infty$

$$\Rightarrow \quad x = c(2-\xi)t + \xi$$

$$= U_{max}\left(1 - \frac{2(2-\xi)}{\rho_j}\right)t + \xi$$

$$= U_{max}\left(1 - \frac{4}{\rho_j}\right)t + \xi\left(1 + \frac{4U_{max}}{\rho_j}t\right)$$

$$\xi = \frac{x - U_{max}\left(1 - \frac{4}{\rho_j}\right)t}{1 + \frac{4U_{max}}{\rho_j}t}$$

$$\rho = \rho_0(\xi) = 2 - \xi = 2 - \frac{x - U_{max}\left(1 - \frac{4}{\rho_j}\right)t}{1 + \frac{4U_{max}}{\rho_j}t}$$

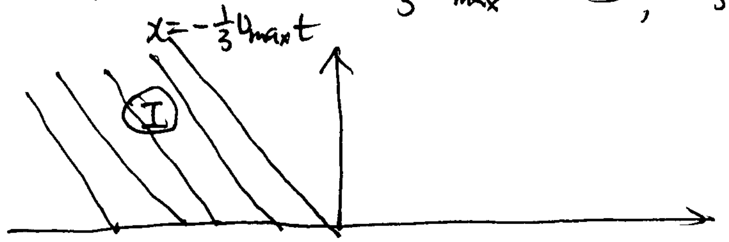
(40pts)

(b) There are three regions.

④

b1) $\xi < 0$, $\rho = \rho_0 = \frac{2\rho_j}{3}$

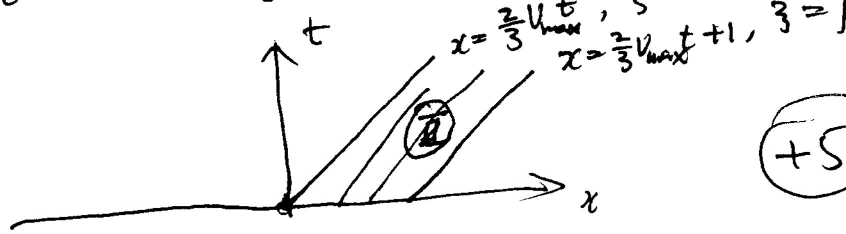
$$x = c\left(\frac{2\rho_j}{3}\right)t + \xi = -\frac{1}{3}U_{max}t + \xi, \quad \xi < 0$$



+5

b2) $0 < \xi < 1$, $\rho = \rho_0 = \frac{\rho_j}{6}$

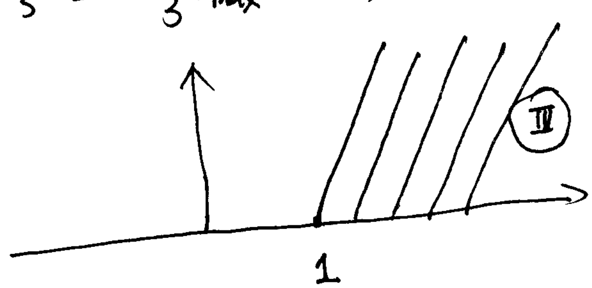
$$x = c\left(\frac{\rho_j}{6}\right)t + \xi = \frac{2}{3}U_{max}t + \xi, \quad 0 < \xi < 1$$



+5

b3) $\xi > 1$, $\rho = \rho_0 = \frac{\rho_j}{3}$

$$x = c\left(\frac{\rho_j}{3}\right)t + \xi = \frac{1}{3}U_{max}t + \xi, \quad \xi > 1$$



+5

between I and II, we use an expansion fan:

$$u = H\left(\frac{x}{t}\right), \text{ where } c(H(\Lambda)) = \Lambda, \quad \Lambda = \frac{x}{t}$$

+5

so $U_{max}\left(1 - \frac{2H}{\rho_j}\right) = \Lambda \Rightarrow H = \frac{\rho_j}{2}\left(1 - \frac{\Lambda}{U_{max}}\right)$

So $u = \frac{\rho_j}{2}\left(1 - \frac{x}{t\,u_{max}}\right)$ for $-\frac{1}{3}u_{max}t < x < \frac{2}{3}u_{max}t$

Between (II) and (III), we have a shock

$$\frac{ds}{dt} = \frac{[Q]}{[\rho]} = \frac{u_{max}\left(1 - \frac{1}{\rho_j}\cdot\frac{1}{3}\rho_j\right)\frac{1}{3}\rho_j - u_{max}\frac{\rho_j}{6}\left(1 - \frac{1}{\rho_j}\frac{1}{6}\rho_j\right)}{\frac{1}{3}\rho_j - \frac{\rho_j}{6}}$$

$$= \frac{1}{2}u_{max}$$

+5

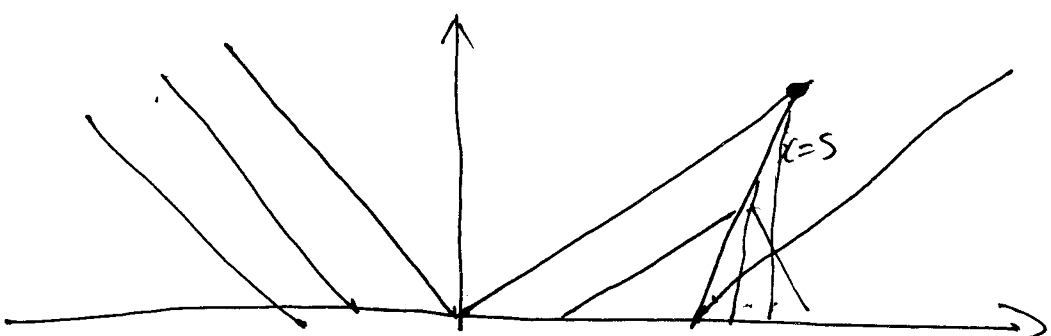
$s(0) = 1 \quad \Rightarrow \quad s = \frac{1}{2}u_{max}t + 1$

The shock curve is $x = \frac{1}{2}u_{max}t + 1$

The expansion fan (last) hits the shock curve at

$$\begin{cases} x = \frac{2}{3}u_{max}t \\ x = \frac{1}{2}u_{max}t + 1 \end{cases} \Rightarrow t = \frac{6}{u_{max}}, \quad x = 4$$

+5



Afterwards, we have

$$\rho_- = \frac{\rho_j}{2}\left(1 - \frac{x}{t\,u_{max}}\right) = \frac{\rho_j}{2}\left(1 - \frac{s}{t\,u_{max}}\right)$$

$$\rho_+ = \frac{\rho_j}{3}$$

+3

So

$$\frac{ds}{dt} = \frac{[Q]}{[\rho]} = \frac{U_{max}\,\rho_-\left(1-\frac{\rho_-}{\rho_j}\right) - U_{max}\,\rho_+\left(1-\frac{\rho_+}{\rho_j}\right)}{\rho_- - \rho_+}$$

$$= U_{max} - \frac{U_{max}}{\rho_j}(\rho_- + \rho_+)$$

+5

$$= U_{max} - \frac{U_{max}}{\rho_j}\left(\frac{\rho_j}{2}\left(1-\frac{s}{tU_{max}}\right) + \frac{\rho_j}{3}\right)$$

$$= \frac{1}{6}U_{max} + \frac{s}{2t} \Rightarrow s = \frac{1}{3}U_{max}t + ct^{\frac{1}{2}}$$

$$s\left(\frac{6}{U_{max}}\right) = 4$$

So $4 = \frac{1}{3}U_{max}\cdot\frac{6}{U_{max}} + c\left(\frac{6}{U_{max}}\right)^{\frac{1}{2}}$

+2

$$c = 2\left(\frac{U_{max}}{6}\right)^{\frac{1}{2}}$$

The shock curve afterwards is

$$s = \frac{1}{3}U_{max}t + 2\left(\frac{U_{max}}{6}\right)^{\frac{1}{2}} t^{\frac{1}{2}}$$

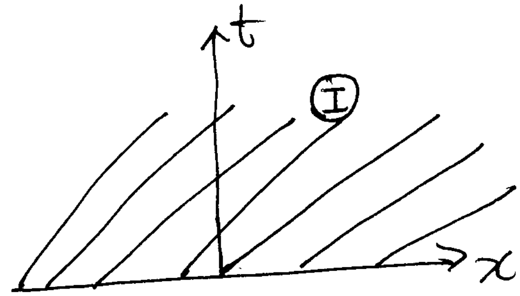
(30 pts)

Problem 2 c) : ~~the g~~ First for $\rho(x,0)=\frac{\rho_j}{8}$, we solve $\boxed{7}$

$$\begin{cases} \frac{dx}{dt}=c(\rho), & x(0)=\xi, \quad -\infty<+\infty \\ \frac{d\rho}{dt}=0, & \rho=\rho_0=\frac{\rho_j}{8} \end{cases}$$

$$x = c(\rho_0)t+\xi$$
$$= c\left(\frac{\rho_j}{8}\right)t+\xi$$
$$= u_{max}\frac{3}{4}t+\xi$$

+5



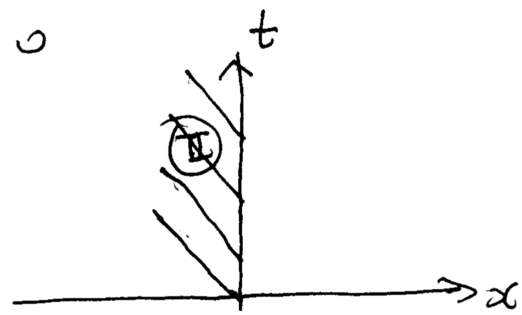
2nd for $\rho(0-,t)=\rho_j$,

$$\begin{cases} \frac{dt}{dx}=\frac{1}{c(\rho)}, & t(0)=\xi, \quad \xi>0 \\ \frac{d\rho}{dt}=0, & \rho=\rho_j \end{cases}$$

$$t=\frac{1}{c(\rho_j)}\cancel{x}+\xi, \quad \rho=\rho_j, \quad \xi>0$$

$$t=\frac{1}{-u_{max}}x+\xi, \quad \xi>0, \quad \rho=\rho_j$$

+5



3rd for $\rho(0+,t)=\frac{\rho_j}{4}$

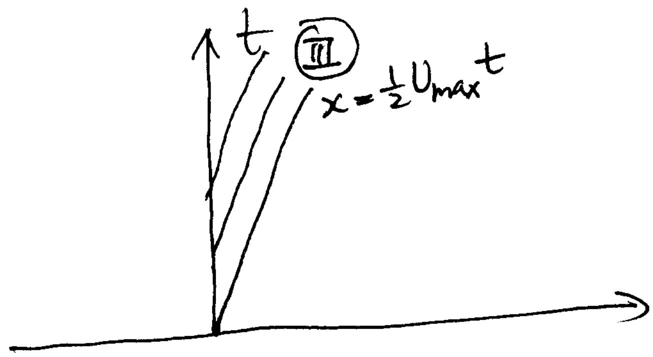
$$\frac{dt}{dx}=\frac{1}{c(\rho)}, \quad t(0)=\xi, \quad \xi>0$$

$$\frac{d\rho}{dt}=0, \quad \rho=\frac{\rho_j}{4}$$

$$t=\frac{1}{c(\frac{\rho_j}{4})}x+\xi, \quad \rho=\frac{\rho_j}{4}, \quad \xi>0.$$

+5

$$t = \frac{1}{\frac{1}{2} U_{max}} x + \xi, \quad \rho = \frac{\rho_j}{4}, \quad \xi > 0$$



Between I and II, there is a shock curve

$$\frac{ds}{dt} = \frac{[Q]}{[\rho]} = \frac{Q(\frac{\rho_j}{8}) - Q(\rho_j)}{\frac{\rho_j}{8} - \rho_j} = -\frac{1}{8} U_{max}$$

$$s(0) = 0$$

$$s = -\frac{1}{8} U_{max} t$$

(+5)

Between III and I, there is an expansion fan

$$u = H\left(\frac{x}{t}\right), \quad c(H(\lambda)) = \lambda, \quad \lambda = \frac{x}{t}$$

$$u = \frac{\rho_j}{2}\left(1 - \frac{\lambda}{U_{max}}\right) = \frac{\rho_j}{2}\left(1 - \frac{x}{t U_{max}}\right)$$

(+5)

So the solution is:

$$u(x,t) = \begin{cases} \frac{\rho_j}{8}, & x < -\frac{1}{8} U_{max} t \\ \rho_j, & -\frac{1}{8} U_{max} t < x < 0 \\ \frac{\rho_j}{4}, & 0 < x < \frac{1}{2} U_{max} t \\ \frac{\rho_j}{2}\left(1 - \frac{x}{t U_{max}}\right), & \frac{1}{2} U_{max} t < x < \frac{3}{4} U_{max} t \\ \frac{\rho_j}{8}, & x > \frac{3}{4} U_{max} t \end{cases}$$

(+5)