Solutions to HW# 3

1 (a) $2u_{xy} + u_{yy} = 0$ (6 pts)

$$2\partial_x\partial_y + \partial_y^2 = (\partial_y + \partial_x)^2 - \partial_x^2$$

Let $\begin{cases} \partial_\xi = \partial_y + \partial_x \\ \partial_\eta = \partial_x \end{cases}$ Then $\begin{cases} \partial_x = \partial_\eta \\ \partial_y = \partial_\xi - \partial_\eta \end{cases}$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$$

So under the change of variable

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

the equation becomes

$$u_{\xi\xi} - u_{\eta\eta} = 0$$

(6 pts)

(b) $\partial_x^2 - 6\partial_x\partial_y + 10\partial_y^2 + \partial_y$

$$= (\partial_x - 3\partial_y)^2 + \partial_y^2 + \partial_y$$

Let $\begin{cases} \partial_\xi = \partial_x - 3\partial_y \\ \partial_\eta = \partial_y \end{cases}$ $\Rightarrow$ $\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$

So under the change of variable

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

the equation becomes $u_{\xi\xi} + u_{\eta\eta} + u_\eta = 0$

(c) $\partial_{x}^2 - 4\partial_x\partial_y + 4\partial_y^2 - 2\partial_x + 3\partial_y$ (8pts)

$= (\partial_x - 2\partial_y)^2 - (2\partial_x - 3\partial_y)$

Let $\begin{cases} \partial_\xi = \partial_x - 2\partial_y \\ \partial_\eta = 2\partial_x - 3\partial_y \end{cases} \Rightarrow \begin{cases} \partial_x = 2\partial_\eta - 3\partial_\xi \\ \partial_y = \partial_\eta - 2\partial_\xi \end{cases}$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$$

So under the change of variable

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

the equation becomes

$$u_{\xi\xi} - u_\eta = 0$$

2. By d'Alembert's Formula

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$
$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

Here $\phi = x^2$, $\psi = 1+x$, $f = xt$

So

$$u(x,t) = \frac{1}{2}\left[(x-ct)^2+(x+ct)^2\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}(1+s)\,ds$$

$$+ \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} y\,dy\right) s\,ds$$

$$= x^2+c^2t^2 + \frac{1}{2c}\left[2ct + \frac{1}{2}\left((x+ct)^2-(x-ct)^2\right)\right]$$

$$+ \frac{1}{2c}\int_0^t s\,\frac{1}{2}\left[(x+c(t-s))^2-(x-c(t-s))^2\right]ds$$

$$= x^2+c^2t^2+t+\cancel{x}t$$

$$+ \frac{1}{6}xt^3$$

Problem3. (a). The well-posedness means:

1) Existence: Given $\phi(x)$, there exists a solution

2) Uniqueness: Given $\phi(x)$, the solution is unique~~ness~~

3) stability: Suppose ϕ is small in some norm
Then u is also small

(b). First we check that $u(x,t)$ satisfies

$$\begin{cases} u_t + k u_{xx} = 0, \ t>0, & -\infty < x < +\infty \\ u(x,0) = \frac{1}{n}\sin nx \end{cases}$$

We note that

$$\max|\phi(x)| = \max|u(x,0)| = \frac{1}{n}\max|\sin nx| = \frac{1}{n}$$

while for $0 \le t \le T$

$$\max|u(x,t)| = \frac{1}{n}e^{n^2\kappa T}\max|\sin nx| = \frac{1}{n}e^{n^2\kappa T}$$

As $n \to +\infty$, $\max|\phi(x)| \to 0$ but $\max\limits_{0\le t\le T}|u(x,t)| \to +\infty$

So this violates the stability criteria.

The problem is not well-posed

Problem 4. (a) Following the wave equation, we decompose the operator into two first order operators

$$2\partial_{tt}^2 + 5\partial_t\partial_x - 3\partial_x^2 = (2\partial_t - \partial_x)(\partial_t + 3\partial_x)$$

Let $\begin{cases} \partial_\xi = 2\partial_t - \partial_x \\ \partial_\eta = \partial_t + 3\partial_x \end{cases} \Rightarrow \begin{array}{l} \partial_t = \frac{3}{7}\partial_\xi + \frac{1}{7}\partial_\eta \\ \partial_x = \frac{2}{7}\partial_\eta - \frac{1}{7}\partial_\xi \end{array}$

$$\begin{pmatrix}\partial_t \\ \partial_x\end{pmatrix} = \begin{pmatrix}\frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7}\end{pmatrix}\begin{pmatrix}\partial_\xi \\ \partial_\eta\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}\frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7}\end{pmatrix}\begin{pmatrix}t \\ x\end{pmatrix} = \begin{pmatrix}\frac{3t-x}{7} \\ \frac{t+2x}{7}\end{pmatrix}$$

So
$$2u_{tt} + 5u_{tx} - 3u_{xx}$$
$$= u_{\xi\eta} = 0$$

Similar to wave equation the general solution is

$$u = f(\xi) + g(\eta)$$
$$= f\left(\frac{3t-x}{7}\right) + g\left(\frac{t+2x}{7}\right)$$
$$= f\left(-\frac{1}{7}(x-3t)\right) + g\left(\frac{2}{7}\left(x+\frac{1}{2}t\right)\right)$$
$$= \tilde{f}(x-3t) + \tilde{g}\left(x+\frac{1}{2}t\right)$$

for any function $\tilde{f}$, $\tilde{g}$

~~(b)~~ (b) Now we find the particular solution

$$u(x,0) = \phi(x) \Rightarrow \tilde{f}(x) + \tilde{g}(x) = \phi(x) \quad -(1)$$
$$u_t(x,0) = \psi(x) \Rightarrow -3\tilde{f}'(x) + \frac{1}{2}\tilde{g}'(x) = \psi(x) \quad -(2)$$

From (2) $\Rightarrow$ $$-3\tilde{f}(x) + \frac{1}{2}\tilde{g}(x) = \int_0^x \psi(x)\,dx + A \quad -(3)$$

From (1) and (3) we have

$$\tilde{f}(x) = \frac{1}{7}\left(\phi(x) - 2\int_0^x \psi - 2A\right)$$
$$\tilde{g}(x) = \frac{2}{7}\left(3\phi(x) + \int_0^x \psi(x)\,dx + A\right)$$

Hence

$$u = \frac{1}{7}\left(\phi(x-3t) - 2\int_0^{x-3t} \psi - 2A\right)$$

$$+ \frac{2}{7}\left(3\phi(x+\tfrac{1}{2}t) + \int_0^{x+\frac{1}{2}t} \psi + A\right)$$

$$= \frac{1}{7}\left(\phi(x-3t) + 6\phi(x+\tfrac{1}{2}t)\right) + \frac{2}{7}\int_{x-3t}^{x+\frac{1}{2}t} \psi(s)\,ds$$

(b) $\phi = \sin x$, $\psi = \cos x$

$$u = \frac{1}{7}\left(\sin(x-3t) + 6\sin(x+\tfrac{1}{2}t)\right)$$

$$+ \frac{2}{7}\int_{x-3t}^{x+\frac{1}{2}t} \cos s\,ds$$

$$= \frac{1}{7}\left(\sin(x-3t) + 6\sin(x+\tfrac{1}{2}t)\right) + \frac{2}{7}\left(\sin(x+\tfrac{1}{2}t) - \sin(x-3t)\right)$$

$$= -\frac{1}{7}\sin(x-3t) + \frac{8}{7}\sin(x+\tfrac{t}{2})$$