Solutions to Homework Assignment 4

1. ($\cancel{1}$) writing $u_{xy} - 4y\,u_x = 0$ as

$$0\cdot u_{xx} + 2\cdot\tfrac{1}{2}u_{xy} + 0\cdot u_{yy} - 4y\,u_x = 0$$

Matrix $A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$, $\det A = -\frac{1}{4} < 0$

So it is hyperbolic type

(2) Let $u_x = v$. Then v satisfies

— (3 pts)

$$v_y - 4y\,v = 0$$

This is 1st order $\Rightarrow$ $\frac{dv}{v} = 4y\,dy$

— ($\cancel{4}$ pts)

$$\Rightarrow v = \cancel{f}(x)\,e^{2y^2}$$

So $u_x = c(x)\,e^{2y^2}$

$$u = \int c(x)\,e^{2y^2}\,dx + g(y) = f(x)e^{2y^2} + g(y), \quad \forall f, g$$

(5 pts)

2. Solution: write $v(x,t) = u(x,t) - t^2$

$$\begin{cases} v_{tt} = u_{tt} - 2 = c^2 u_{xx} - 2 = c^2 v_{xx} - 2, & x > 0 \\ v(x,0) = x, \quad v_t(x,0) = 0 \end{cases}$$

—($\cancel{2}$3pts)

(2pts)

Extend $f(x,t) = -2$ to odd: $f_{ext}(x,t) = \begin{cases} -2, & x > 0 \\ 2, & x < 0 \end{cases}$

So $$V(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds$$ —(3pts)

$$= \frac{1}{2}\left[x+ct+x-ct\right] + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds$$

For the integral $\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds$, we divide it into two cases:

Case1: $x>ct \Rightarrow x>c(t-s) \Rightarrow x-c(t-s)>0$, $f_{ext}=-2$

$$\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}(-2)\,dy\,ds = \frac{1}{2c}\int_0^t(-4c)(t-s)\,ds = -t^2$$

— (5pts)

So $u(x,t)=V(x,t)+t^2=x$

Case2: $x<ct$. In this case $x-c(t-s)=0 \Leftrightarrow s=t-\frac{x}{c}$

$$\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds$$

$$=\frac{1}{2c}\int_0^{t-\frac{x}{c}}\left(\int_{x-c(t-s)}^{0} f_{ext}+\int_0^{x+c(t-s)} f_{ext}\right)+\frac{1}{2c}\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds$$

$$=\frac{1}{2c}\left[\int_0^{t-\frac{x}{c}}\left(2(x-c(t-s))\right)\right]+\frac{1}{2c}\int_0^{t-\frac{x}{c}}(-2)(x+c(t-s))+\frac{1}{2c}(-2)\int_{t-\frac{x}{c}}^{t}(2c(t-s))\,ds$$

$$=\frac{1}{2}\left(t-\frac{x}{c}\right)^2-\frac{1}{c}\left((x+ct)\left(t-\frac{x}{c}\right)-\frac{c}{2}\left(t-\frac{x}{c}\right)^2\right)-\frac{1}{c}\cdot\left(\frac{x}{c}\right)^2$$

(8pts)

$$=\left(t-\frac{x}{c}\right)^2-t^2$$

So $u(x,t)=V(x,t)+t^2=x+\left(t-\frac{x}{c}\right)^2$

Another method: Write $V(x,t) = u(x,t) - x$

Then
$$\begin{cases} V_{tt} = c^2 V_{xx} \\ V(x,0) = V_t(x,0) = 0, \quad x>0 \\ V(0,t) = t^2 \end{cases}$$
(5 pts)

The Solutions to $V_{tt} = c^2 V_{xx}$ is given by

$$V(x,t) = f(x+ct) + g(x-ct).$$
(3 pts)

From the initial condition
$V(x,0) = V_t(x,0) = 0 \Rightarrow$

$f(x) = g(x) = 0$ for $x>0$.
(3 pts)

From the boundary condition $\Rightarrow$

$$f(ct) + g(-ct) = t^2$$
$$s = ct, \quad f(s) + g(-s) = \left(\frac{s}{c}\right)^2$$
(6 pts)

So $s>0$, $f(s) = 0$, $g(-s) = \left(\frac{s}{c}\right)^2 \Rightarrow g(\bar{s}) = \left(\frac{\bar{s}}{c}\right)^2$, for $\bar{s}<0$

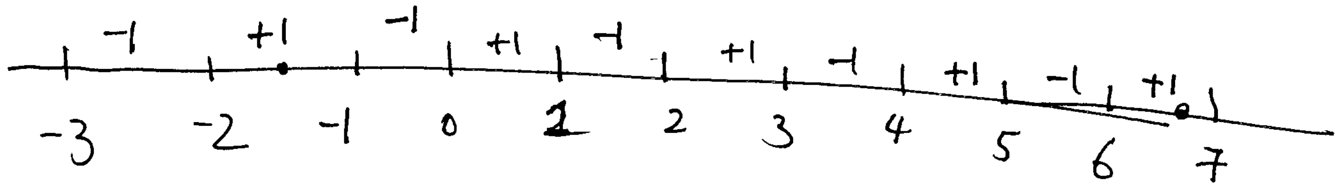
Thus
$$V(x,t) = f(x+ct) + g(x-ct)$$
$$= 0 + \begin{cases} 0, & x>ct \\ \left(\frac{x-ct}{c}\right)^2 = \left(t-\frac{x}{c}\right)^2, & x<ct. \end{cases}$$
(3 pts)

Problem3: $\phi(x) = 1 = \psi(x)$. Extend ϕ and ψ oddly to $(-1, 0)$ and periodically to $\mathbb{R}$ with period 2.

4

$\phi_{ext} = \psi_{ext} =$

(5 pts)



Here $x = \frac{5}{2}$, $c = 2$, $t = 2$

So $x + ct = \frac{5}{2} + 4 = 6.5$ (5 pts)

$x - ct = \frac{5}{2} - 4 = -1.5$

By d'Alembert's Formula,

$$u\left(\tfrac{5}{2}, 2\right) = \frac{1}{2}\left[\phi_{ext}(6.5) + \phi_{ext}(-1.5)\right] + \frac{1}{4}\int_{-1.5}^{6.5} \psi_{ext}$$

$$= \frac{1}{2}\left[1 + 1\right] + \frac{1}{4}\left[\int_{-1.5}^{-1} 1 + \int_{-1}^{0} (-1) + \int_{0}^{1} (1) + \int_{1}^{2} -1 + \int_{2}^{3} +1 + \int_{3}^{4} -1 + \int_{4}^{5} +1 + \int_{5}^{6} -1 + \int_{6}^{6.5} 1\right]$$

$$= \frac{1}{2}\left[1 + 1\right] + \frac{1}{4}\left[0.5 - 1 + 0.5\right]$$

$$= 1$$

(10 pts)

If you use separation of variables and if your answer is correct, you also get full marks

Problem 4. Let $V(x,t) = u(x,t) - 1$. Then

⑤

$$\begin{cases} V_t = k V_{xx} \\ V(x,0) = -1 \\ V(0,t) = 0 \end{cases} \qquad (5\text{pts})$$

Extend ϕ oddly to (8pts)

$$\phi_{odd} = \begin{cases} -1, & x > 0 \\ 1, & x < 0 \end{cases}$$

$$V(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\,\phi_{odd}(y)\,dy = \int_0^{+\infty} \big(S(x-y,t) - S(x+y,t)\big)\,\phi(y)\,dy$$

$$= \int_0^{+\infty} \big(S(x+y,t) - S(x-y,t)\big)\,dy$$

$$\int_0^{+\infty} S(x+y,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^{+\infty} e^{-\frac{(x+y)^2}{4kt}}\,dy \xrightarrow{y+x = \sqrt{4kt}\,p} \frac{1}{\sqrt{4k\pi t}} \int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-p^2}\sqrt{4kt}\,dp$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-p^2}\,dp \qquad (5\text{pts})$$

$$\int_0^{+\infty} S(x-y,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^{+\infty} e^{-\frac{(x-y)^2}{4kt}}\,dy \xrightarrow{y-x = \sqrt{4kt}\,p} \frac{1}{\sqrt{4k\pi t}} \int_{\frac{-x}{\sqrt{4kt}}}^{+\infty} e^{-p^2}\sqrt{4kt}\,dp$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4kt}}}^{+\infty} e^{-p^2}\,dp \qquad (5\text{pts})$$

$$V(x,t) = \frac{1}{\sqrt{\pi}}\left(\int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-p^2} - \int_{\frac{-x}{\sqrt{4kt}}}^{+\infty} e^{-p^2}\,dp \right) = -\frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4kt}}}^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp = \frac{-2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp$$

(3pts)

$$u(x,t) = -\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp + 1$$

Problem 5: Suppose $u_1(x,t)$, $u_2(x,t)$ satisfy (6)

$$\begin{cases} u_t = k u_{xx} + f(x,t) \\ u(x,0) = \phi(x) \\ u_x(0,t) - a_0 u(0,t) = g_1(t) \\ u_x(l,t) + a_1 u(l,t) = g_2(t) \end{cases}$$

5 pts

Then let $u(x,t) = u_1(x,t) - u_2(x,t)$

$$u_t - k u_{xx} = u_{1,t} - k u_{1,xx} - (u_{2,t} - k u_{2,xx})$$
$$= f(x,t) - f(x,t) = 0$$

$$u(x,0) = u_1(x,0) - u_2(x,0) = 0$$
$$u_x(0,t) - a_0 u(0,t) = u_{1,x}(0,t) - a_0 u_1(0,t) - (u_{2,x}(0,t) - a_0 u_2(0,t))$$
$$= g_1(t) - g_1(t) = 0$$

5pts

Similarly
$$u_x(l,t) + a_1 u(l,t) = 0$$

Now let $E(t) = \frac{1}{2}\int_0^l u^2(x,t)\,dx$. Then

$$\frac{dE}{dt} = \int_0^l u\, u_t = \int_0^l u(k u_{xx}) = k\int_0^l ((u u_x)_x - u_x^2)$$
$$= k u u_x\Big|_0^l - k\int_0^l u_x^2$$
$$= k u(l,t) u_x(l,t) - k u(0,t) u_x(0,t) - k\int_0^l u_x^2$$
$$= -a_1 k u^2(l,t) - k a_0 u^2(0,t) - k\int_0^l u_x^2 \le 0$$

5pts

So $E(t) \le E(0) = \frac{1}{2}\int_0^l u^2(x,0)\,dx = 0 \Rightarrow \int_0^l u^2(x,t)\,dx \le 0$

$\Rightarrow u(x,t) \equiv 0$

$\Rightarrow u_1(x,t) = u_2(x,t)$

This proves uniqueness.

5pts