Solutions to HW#5

Problem 1: This is a standard separation of variable problem

Step 1: $u = X(x)T(t)$

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda$$

EVP : $X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(\ell) = 0$ — (2 pts)

ODE : $T'' + c^2 \lambda T = 0$ (2 pts)

Step 2: Solve EVP: λ is positive

$$\lambda = \beta^2 \Rightarrow X = c_1 \cos\beta x + c_2 \sin\beta x$$

$$X'(0) = 0 \Rightarrow c_2 = 0$$

$$X(\ell) = 0 \Rightarrow \cos\beta\ell = 0, \quad \beta\ell = (n - \tfrac{1}{2})\pi, \ n = 1, 2, \cdots$$

$$\lambda_n = \beta_n^2 = \left(\frac{(n-\frac{1}{2})\pi}{\ell}\right)^2, \quad n = 1, 2 \cdots$$

$$X_n(x) = \cos\beta_n x$$

6 pts

$$T'' + c^2\lambda T = T'' + (c\beta_n)^2 T = 0$$

$$T = a\cos(c\beta_n t) + b\sin(c\beta_n t)$$

+ 2 pts

Step 3: Sum up

$$u(x,t) = \sum_{n=1}^{+\infty} \left(a_n \cos(c\beta_n t) + b_n \sin(c\beta_n t)\right)\cos\beta_n x$$ — 2 pts

$$u(x,0) = 0 \Rightarrow \sum_{n=1}^{+\infty} a_n \cos\beta_n x = 0 \Rightarrow a_n = 0$$ 2 pts

$$u_t(x,0) = x \Rightarrow \sum_{n=1}^{+\infty} (c\beta_n b_n)\cos\beta_n x = x \Rightarrow c\beta_n b_n = \frac{\int_0^\ell x\cos\beta_n x\,dx}{\int_0^\ell \cos^2\beta_n x\,dx}$$ — 2 pts

Here

$$\int_0^l \cos^2 \beta_n x \, dx = \int_0^l \frac{1+\cos(2\beta_n x)}{2} dx = \frac{l}{2}$$

$$\int_0^l x \cos\beta_n x \, dx = \frac{x}{\beta_n} \sin\beta_n x + \frac{1}{\beta_n^2} \cos\beta_n x \Big|_0^l$$

$$= \frac{l}{\beta_n} \sin\left((n-\tfrac{1}{2})\pi\right) + \frac{1}{\beta_n^2} \cos\left((n-\tfrac{1}{2})\pi\right) - \frac{1}{\beta_n^2}$$

$$= \frac{l}{\beta_n} (-1)^{n+1} - \frac{1}{\beta_n^2}$$

2pts

So

$$b_n = \frac{2}{(c\beta_n) l} \left[\frac{l}{\beta_n} (-1)^{n+1} - \frac{1}{\beta_n^2} \right]$$

Final solution is

$$u(x,t) = \sum_{n=1}^{+\infty} \frac{2}{(c\beta_n l)} \left[\frac{l}{\beta_n} (-1)^{n+1} - \frac{1}{\beta_n^2} \right] \sin(c\beta_n t) \cos\beta_n x$$

Problem 2. Step 1: $u = X(x) T(t)$

$$\frac{X''(x)}{X(x)} = \frac{T'}{kT} = -\lambda$$

$X'' + \lambda X = 0, \quad X(0) = X(1), \quad X'(0) = X'(1).$ — 2pts

$T' + \lambda k T = 0$ — 2pts

Step2: Solve (EVP):

$\lambda_0 = 0 \Rightarrow X_0 = 1$ — 2pts

$\lambda_n = \beta_n^2 \Rightarrow X(x) = c_1 \cos\beta_n x + c_2 \sin\beta_n x$

$X(0) = X(1), \quad X'(0) = X'(1) \Rightarrow \sin\beta_n x = 1 - \cos\beta_n x = 0$ | 4pts

$\beta_n = 2n\pi$

$X_n(x) = a_1 \cos(2n\pi x) + b \sin(2n\pi x)$

ODE: $\lambda_0 = 0 \Rightarrow T = c$

$\lambda_n = \beta_n^2 \Rightarrow T' + k\beta_n^2 T = 0 \Rightarrow T = a\, e^{-k\beta_n^2 t}$ | 2pts

Step3: Sum up

$$u = a_0 + \sum_{n=1}^{+\infty} \left(a_n \cos(2n\pi x) + b_n \sin(2n\pi x)\right) e^{-k(2n\pi)^2 t}$$ | 2pts

$$u(x,0) = \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$a_0 = \int_0^1 \phi(x)\,dx = \frac{1}{2}\int_0^1 (1 + \cos 2x) = \frac{1}{2} + \frac{1}{4}\sin 2$$ | 2pts

$$a_n = \frac{\int_0^1 \phi(x)\cos(2n\pi x)\,dx}{\int_0^1 \cos^2(2n\pi x)\,dx} = 2\cdot\int_0^1 \frac{1}{2}(1+\cos 2x)\cos(2n\pi x)\,dx$$

$$= \int_0^1 \cos 2x \cos(2n\pi x)\,dx = \int_0^1 \frac{1}{2}\left[\cos(2n\pi+2)x + \cos(2n\pi-2)x\right] dx$$

$$= \frac{1}{4n\pi+4}\sin 2 + \frac{1}{4n\pi-4}\sin 2$$ | 2pts

$$b_n = \frac{\int_0^1 \phi(x)\sin(2n\pi x)\,dx}{\int_0^1 \sin^2(2n\pi x)\,dx}$$

$$= \int_0^1 (1+\cos(2x))\,\sin(2n\pi x)\,dx$$

$$= \int_0^1 \sin(2n\pi x)\cos 2x\,dx$$

$$= \frac{1}{2}\int_0^1 \left(\sin(2n\pi+2)x - \sin(2n\pi-2)x\right)dx$$

$$= \frac{1}{2}\left(\frac{1}{2n\pi+2}(1-\cos 2) - \frac{1}{2n\pi-2}(1-\cos 2)\right)$$

$$= (1-\cos 2)\left(\frac{1}{4n\pi+4} - \frac{1}{4n\pi-4}\right) \quad \text{— 2pts}$$

Problem 3. (a). Step 1. $u = X(x)T(t)$

$$\frac{X''}{X} = -\frac{T'}{T} = -\lambda$$

$$X'' + \lambda X = 0, \quad T' + \lambda T = 0$$

$$X'(0) + X(0) = 0, \quad X'(1) + 2X(1) = 0 \quad \text{(2pts)}$$

Step 2. Solve EVP & ODE

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < 1 \\ X'(0) + X(0) = 0, & X'(1) + 2X(1) = 0 \end{cases}$$

Case 1. $\lambda = -\gamma^2 < 0, \quad \gamma > 0$

$$X = c_1\cosh\gamma x + c_2\sinh\gamma x. \quad \text{(2pts)}$$

$$X' = \gamma c_1\sinh\gamma x + \gamma c_2\cosh\gamma x$$

$X'(0)+X(0)=0 \Rightarrow$

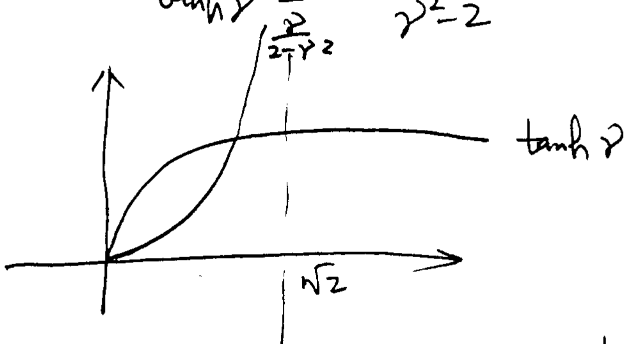
$$\gamma c_2 + c_1 = 0 \Rightarrow c_1 = -\gamma c_2 \Rightarrow c_2 = -\frac{1}{\gamma} c_1$$

$X'(1)+2X(1)=0$

$$\gamma c_1 \sinh\gamma + \gamma c_2 \cosh\gamma + 2(c_1\cosh\gamma + c_2\sinh\gamma) = 0$$

$$-\gamma^2 c_2 \sinh\gamma + \gamma c_2 \cosh\gamma + 2(-c_2\gamma\cosh\gamma + c_2\sinh\gamma) = 0$$

$$\tanh\gamma = -\frac{\gamma}{\gamma^2-2} = \frac{\gamma}{2-\gamma^2} \qquad (4\text{pts})$$



$$(\tanh\gamma)'\Big|_{\gamma=0} = 1, \quad \left(\frac{\gamma}{2-\gamma^2}\right)'\Big|_{\gamma=0} = \frac{1}{2} \qquad (2\text{pts})$$

So $\exists$ a unique negative eigenvalue, called γ_0

$$X_0(x) = \cosh(\gamma_0 x) - \frac{1}{\gamma_0}\sinh\gamma_0 x$$

Case 2. $\lambda = 0 \Rightarrow X = c_1 + c_2 x$ (2pts)

$X' = c_2$

$$\left.\begin{array}{l} X'(0)+X(0)=0 \Rightarrow c_2+c_1=0 \\ X'(1)+2X(1)=0 \Rightarrow c_2+2c_1=0 \end{array}\right\} \Rightarrow c_1=c_2=0$$

Case 3. $\lambda = \beta^2,\ \beta>0 \Rightarrow X = c_1\cos\beta x + c_2\sin\beta x$

we have

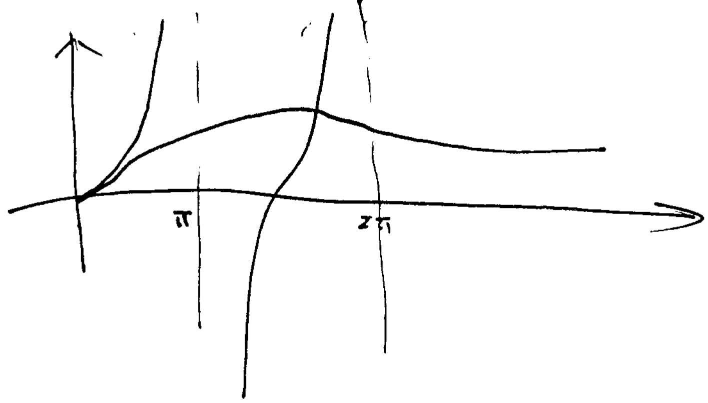
$X'(0)+X(0)=0 \Rightarrow -\beta c_2 + c_1 = 0 \Rightarrow c_1 = \beta c_2$

$X'(1)+2X(1)=0$ 2pts

$\Rightarrow \beta c_1 \sin\beta - \beta c_2 \cos\beta$

$+2(c_1\cos\beta + c_2 \sin\beta) = 0$

$\Rightarrow \tan\beta = \dfrac{\beta}{\beta^2+2}$ (4pts)



$n\pi < \beta_n < (n+1)\pi, \quad n=1,2,\cdots$ (2pts)

$X_n = \cos\beta_n x + \frac{1}{\beta_n}\sin\beta_n x$ (2pts)

Now corresponding to λ_n, we have (2pts)

$T_n(t) = c\, e^{-\lambda_n t}$

Step3 Sum up

$u(x,t) = \sum_{n=0}^{+\infty} a_n e^{-\lambda_n t} X_n(x)$ (2pts) where

$$a_n = \frac{\int_0^1 \phi(x) X_n(x)\,dx}{\int_0^1 X_n^2(x)\,dx}, \quad n=0,1,2,3,\dots \quad (4\text{pts})$$

(b) $u(x,t) = a_0 e^{\gamma_0^2 t} X_0(x) + \sum_{n=1}^{+\infty} a_n e^{-\beta_n^2 t} X_n(x)$

Since all the other terms contain the term $e^{-\beta_n^2 t}$, and hence they are bdd. (5pts)

then if $u(x,t)$ is bdd, then $a_0 = 0$

Hence u is bdd if and only if $a_0 = 0$ (5pts)

$$\Leftrightarrow \int_0^1 \phi(x)\left(\cosh \gamma_0 x - \frac{1}{\gamma_0}\sinh \gamma_0 x\right) dx = 0$$

Problem 4. Let $X = X_R + i X_I$

$$\lambda = \lambda_R + i\lambda_I$$

Then

$$0 = X'' + \lambda X = X_R'' + i X_I'' + (\lambda_R + i\lambda_I)(X_R + i X_I)$$

$$= X_R'' + \lambda_R X_R - \lambda_I X_I + i\left(X_I'' + \lambda_R X_I + \lambda_I X_R\right)$$

5 pts

So

$$X_R'' + \lambda_R X_R - \lambda_I X_I = 0 \qquad (1)$$

$$X_I'' + \lambda_R X_I + \lambda_I X_R = 0 \qquad (2)$$

Multiplying (1) by X_I and (2) by X_R and subtracting and integrating

5pts

$$\int_0^l \left(X_I X_R'' - X_R X_I''\right) - \lambda_I \int_0^l \left(X_I^2 + X_R^2\right) = 0$$

(7pts)

By Lagrange's identity

$$\int_0^l X_I X_R'' - X_R X_I'' = \int_0^l \left(X_I X_R' - X_I' X_R\right)'$$

$$= X_I X_R' - X_I' X_R \Big|_0^l$$

$$= X_I(l) X_R'(l) - X_I'(l) X_R(l) - \left(X_I(0) X_R'(0) - X_I'(0) X_R(0)\right)$$

$$\cancel{(5X(0) + X(0))}$$

$$= X_I(0)\left(5X_R(0) + X_R'(0)\right) - \left(5X_I(0) + X_I'(0)\right) X_R(0) - \left(X_I(0) X_R'(0) - X_I'(0) X_R\right.$$

$$= 0$$

Hence we obtain

$$\lambda_I \int_0^\ell (x_I^2 + x_R^2) = \int_0^\ell (x_I x_R'' - x_R x_I'') \qquad (3\text{pts})$$

$$= 0 .$$

Thus

$$\lambda_I = 0 \qquad (1\text{pts})$$

So $\lambda = \lambda_R$ is real.

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