Solutions to Homework #6

Problem 1:

(a) $x^2\phi_{xx}+5x\phi_x+\lambda\phi=0$, $1\le x\le 2$, $\phi(1)=\phi(2)=0$

Sol'n: Write

$$\phi_{xx}+\frac{5}{x}\phi_x+\frac{\lambda}{x^2}\phi=0$$

~~Multiplying~~ by p and we want

$$\frac{p'}{p}=\frac{5}{x} \Rightarrow p=x^5$$

So $$x^5\phi_{xx}+5x^4\phi_x+\lambda x^3\phi$$

$$(x^5\phi_x)_x+\lambda x^3\phi=0,\ 1\le x\le 2,$$

$p(x)=x^5$, $q(x)=0$, $w(x)=x^3$ — (5pts)

Eigenfunction: Try $\phi(x)=x^r$

$$r(r-1)+5r+\lambda=0$$

$$r^2+4r+\lambda=0$$

$$(r+2)^2+\lambda-4=0 \quad \text{— 2pts}$$

Case 1 $\lambda<4 \Rightarrow r=-2\pm\beta$, $\beta=\sqrt{4-\lambda}$

~~$x^r=x^{2\pm\beta i}=x^2\cdot x^{\beta i}=x^2e^{i\beta\ln x}=x^2\cos(\beta\ln x)+x^2$~~

$\phi(x)=c_1x^{-2-\beta}+c_2x^{-2+\beta}$

$\phi(1)=\phi(2)=0 \Rightarrow \left.\begin{array}{l} c_1+c_2=0 \\ c_1 2^{-2-\beta}+c_2 2^{-2+\beta}=0\end{array}\right\} \Rightarrow c_1=c_2=0$ — 2pts

Case 2. $\lambda = 4$

$\phi_1(x) = x^{-2}$, $\phi_2(x) = x^{-2}\ln x$

$\phi = c_1 x^{-2} + c_2 x^{-2}\ln x$

$\phi(1) = 0 \Rightarrow c_1 + c_2 \ln 1 = 0 \Rightarrow c_1 = 0$

$\phi(2) = 0 \Rightarrow c_1 2^{-2} + c_2 2^{-2}\ln 2 = 0 \Rightarrow c_2 = 0$

1pts

Case 3. $\lambda > 4$

$r = -2 \pm \beta i$, $\beta = \sqrt{\lambda - 4}$

$\phi(x) = x^{-2+\beta i} = x^{-2} \cdot x^{\beta i} = x^{-2} e^{i\beta \ln x}$

$= x^{-2}(\cos \beta \ln x + i \sin \beta \ln x)$

So $\phi(x) = c_1 x^{-2} \cos \beta \ln x + c_2 x^{-2} \sin \beta \ln x$

$\phi(1) = 0 \Rightarrow c_1 = 0$

$\phi(2) = 0 \Rightarrow c_2 2^{-2} \sin(\beta \ln 2) = 0 \Rightarrow \beta \ln 2 = n\pi$, $n = 1, 2, \cdots$

Thus $\beta = \sqrt{\lambda - 4} = \frac{n\pi}{\ln 2} \Rightarrow \lambda = 4 + \left(\frac{n\pi}{\ln 2}\right)^2$

(3pts)

$X_n(x) = x^{-2} \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$

Orthogonality condition is:

$$0 = \int_1^2 w(x) X_n(x) X_m(x)\, dx = \int_1^2 x^3 \left(x^{-2} \sin\left(\frac{n\pi}{\ln 2}\ln x\right) x^{-2} \sin\left(\frac{m\pi}{\ln 2}\ln x\right)\right) dx$$

when $n \neq m$

(-2pts)

(b) $\phi_{xx} - 2\phi_x + \lambda\phi = 0$

sol'n: $\frac{p'}{p} = -2 \Rightarrow p = e^{-2x}$

$$(e^{-2x}\phi_x)_x + \lambda e^{-2x}\phi = 0$$

$p(x) = e^{-2x}$, $q = 0$, $\sigma = e^{-2x}$ — (5 pts)

So λ must be real and positive.

Eigenfunctions: $\phi(x) = e^{rx}$

$$r^2 - 2r + \lambda = 0$$

$$(r-1)^2 = 1-\lambda.$$ — (2 pts)

Case 1 $\lambda < 1$. $\beta^2 = 1-\lambda$, $\beta > 0$

$r = 1 \pm \beta$

$\phi(x) = c_1 e^{(1-\beta)x} + c_2 e^{(1+\beta)x}$.

$\left.\begin{array}{l} \phi(0) = 0 \Rightarrow c_1 + c_2 = 0 \\ \phi(1) = 0 \Rightarrow c_1 e^{(1-\beta)} + c_2 e^{(1+\beta)} = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$

2 pts

Case 2 $\lambda = 1$, $r_1 = r_2 = 1$

$\phi_1(x) = e^x$, $\phi_2(x) = e^x \cdot x$.

$\phi = c_1 e^x + c_2 x e^x$

$\left.\begin{array}{l} \phi(0) = 0 \Rightarrow c_1 = 0 \\ \phi(1) = 0 \Rightarrow c_1 + c_2 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$

1 pt

Case3. $\lambda > 1$

$$(r-1)^2 = 1-\lambda \qquad r = 1 \pm \beta i, \quad \beta = \sqrt{\lambda-1}$$

$$e^{rx} = e^{x}\cos\beta x + i e^{x}\sin\beta x$$

so $\quad \phi = c_1 e^{x}\cos\beta x + c_2 e^{x}\sin\beta x.$

3pt

$\phi(0) = 0 \Rightarrow c_1 = 0$

$\phi(1) = c_2 e^{1}\sin\beta = 0 \quad \Rightarrow \quad \beta = n\pi, \quad n = 1, 2, \cdots$

$\sqrt{\lambda-1} = n\pi \quad \Rightarrow \quad \lambda = (n\pi)^2 + 1$

$X(x) = e^{x}\sin(n\pi x), \quad n = 1, 2, \cdots$

Orthogonality condition is

2pt

$$0 = \int_0^1 \omega(x)\, X_n(x)\, X_m(x)\, dx$$

$$= \int_0^1 e^{-2x}\left(e^{x}\sin(n\pi x)\right)\left(e^{x}\sin(m\pi x)\right) dx = 0$$

Problem 2: We write (eigenvalues $\lambda = n^2$, $X_n(x) = \sin nx$)

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(t)\sin(nx)$$

$$\phi(x) = \sum_{n=1}^{+\infty} \phi_n \sin nx$$

$$u_{tt} = \sum_{n=1}^{+\infty} u_n''(t)\sin(nx)$$

$$\psi(x) = \sum_{n=1}^{+\infty} \psi_n \sin nx$$

$$u_{xx} = \sum_{n=1}^{+\infty} w_n(t)\sin nx$$

$$e^t \sin 3x = \sum_{n=1}^{+\infty} f_n(t)\sin nx \quad \left(\Rightarrow f_3(t) = e^t,\ f_n(t) = 0 \ \forall n \neq 3\right)$$

$$w_n(t) = \frac{2}{\pi}\int_0^\pi u_{xx}\sin nx = \frac{2}{\pi}\int_0^\pi u_{xx}\sin nx + (n^2 \sin nx)u - n^2 \sin nx\, u$$

$$= \frac{2}{\pi}\left(u_x \sin nx + u(\cos nx)\, n\right)\Big|_0^\pi - n^2 u_n(t)$$

$$= \frac{2}{\pi}\left(n \cdot u(\pi,t)\cos n\pi - n\, u(0,t)\cos 0\right) - n^2 u_n(t) \qquad \text{(5pts)}$$

$$= -\frac{2n}{\pi}t - n^2 u_n(t)$$

So

$$\begin{cases} u_n''(t) + n^2 u_n(t) = -\frac{2n}{\pi}t + f_n(t) & \text{— (1)} \\ u_n(0) = \phi_n, \quad u_n'(0) = \psi_n \end{cases} \qquad \text{(2pts)}$$

$f_n(t) = d_n e^t$ So the general sol'n of (1) is

$$u_n(t) = A\cos nt + B\sin nt + c_1 t + c_2 e^t \qquad \text{(5pts)}$$

where $n^2 c_1 = -\frac{2n}{\pi} \Rightarrow c_1 = -\frac{2}{n\pi}$, $\quad c_2 = \frac{d_n}{(1+n^2)}$

$$u_n(0) = \phi_n \Rightarrow A + c_2 = \phi_n \Rightarrow A = \phi_n - c_2$$

$$u_n'(0) = \psi_n \Rightarrow nB + c_1 + c_2 = \psi_n \Rightarrow B = -(c_1 + c_2 - \psi_n)\frac{1}{n}$$

Now for $n \neq 3, 5$: $f_n = 0$, $\phi_n = 0$, $\psi_n = 0$

$$u_n(t) = B \sin nt - \frac{2}{n\pi} t$$

$$= \left(+\frac{2}{n\pi}\right)\frac{1}{n} \sin nt - \frac{2}{n\pi} t$$

2pts

For $n=3$, $f_n = e^t$, $\phi_n = 1$, $\psi_n = 0$

$$A = 1 - \frac{1}{1+n^2}, \quad d_3 = 1$$

$$B = (\psi_n - c_1 - c_2)\frac{1}{n}$$

$$= \left(0 + \frac{2}{n\pi} - \frac{1}{1+n^2}\right)\frac{1}{n}$$

For $n=5$, $f_n = 0$, $\phi_n = 0$, $\psi_n = 1$

$$c_1 = -\frac{2}{n\pi}, \quad c_2 = 0$$

$$A = 0$$

$$B = \left(1 + \frac{2}{n\pi}\right)\frac{1}{n}$$

(5pts)

3. (a). Step 1: $u = X(x)\,Y(y)$

$$\frac{X''}{X} = -\frac{Y''}{Y} = +\lambda$$

5pts

$$\begin{cases} X'' - \lambda X(x) = 0 \\ X(\pi) = 0 \end{cases} \qquad \begin{cases} Y'' + \lambda Y = 0 \\ Y'(0) = 0,\ Y(\pi) = 0 \end{cases}$$

Step 2. $Y = \cos\left(n-\frac{1}{2}\right)y, \quad n = 1, 2, \cdots$

$\lambda = \left(n-\frac{1}{2}\right)^2, \quad n = 1, 2, \cdots$

(5 pts)

$$X'' - \left(n-\tfrac{1}{2}\right)^2 X(x) = 0$$

$$X = A\cosh\left(\left(n-\tfrac{1}{2}\right)(x-\pi)\right) + B\sinh\left(\left(n-\tfrac{1}{2}\right)(x-\pi)\right)$$

$$X(x) = \sinh\left(\left(n-\tfrac{1}{2}\right)(x-\pi)\right)$$

Step 3: $u = \sum_{n=1}^{+\infty} a_n \sinh\left(\left(n-\frac{1}{2}\right)(x-\pi)\right)\cos\left(n-\frac{1}{2}\right)y$,

Now $u(0,y) = \cos^2 y = \frac{1+\cos 2y}{2}$

(5pts)

$$\frac{1+\cos 2y}{2} = \sum a_n \sinh\left(\left(n-\tfrac{1}{2}\right)(-\pi)\right)\cos\left(n-\tfrac{1}{2}\right)y$$

$$-\sinh\left(\left(n-\tfrac{1}{2}\right)\pi\right)a_n = \frac{\int_0^\pi \cos^2 y \cos\left(\left(n-\frac{1}{2}\right)y\right)dy}{\int_0^\pi \cos^2\left(\left(n-\frac{1}{2}\right)y\right)} = \frac{2}{\pi}\int_0^\pi \left(\frac{1+\cos 2y}{2}\right)\left(\cos\left(n-\tfrac{1}{2}\right)y\right)$$

$$= \frac{4}{\pi}\left[\frac{1}{n-\frac{1}{2}}\sin(n-\tfrac{1}{2})\pi + \frac{1}{2}\int_0^{\pi}\left(\cos(2+n-\tfrac{1}{2})y - \cos(n-\tfrac{1}{2}-2)y\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n-\frac{1}{2}}\sin(n-\tfrac{1}{2})\pi + \frac{1}{2}\cdot\left(\frac{1}{n+\frac{3}{2}}\sin(n+\tfrac{3}{2})\pi - \frac{1}{n-\frac{5}{2}}\sin(n-\tfrac{5}{2})\pi\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{1}{n-\frac{1}{2}} + \frac{1}{2}\left(\frac{1}{n+\frac{3}{2}} - \frac{1}{n-\frac{5}{2}}\right)\right)\underbrace{\sin(n-\tfrac{1}{2})\pi}_{(-1)^{n-1}}$$

— (5pts)

(b). Suppose there are two solutions u_1, u_2.

Then let $v(x,y) = u_1(x,y) - u_2(x,y)$.

~~v satisfies~~

$$\Delta v = 0 \quad \text{in } D$$

$$v_y(x,0) = v(x,\pi) = v(\pi,y) = v(0,y) = 0$$

3pts

$$0 = \int_D v\Delta v = \int_D \nabla(v\nabla v) - \int_D |\nabla v|^2$$

$$0 = \int_{\partial D} v\frac{\partial v}{\partial n} - \int |\nabla v|^2$$

$$\int_D |\nabla v|^2 = \int_{\partial D} v\frac{\partial v}{\partial n} = \int_{\{y=0\}} v\left(-\frac{\partial v}{\partial y}\right) + \int_{\{y=\pi\}} v\frac{\partial v}{\partial n} + \int_{\{x=\pi\}} v\frac{\partial v}{\partial n}$$

$$+ \int_{\{x=0\}} v\frac{\partial v}{\partial n} = 0$$

6pts

So $|\nabla v|^2 = 0 \Rightarrow v = \text{Constant} \Rightarrow v = 0$ 1pt

Problem 4. Write it in polar coordinate

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 1 \quad \text{in } \{r<2\}$$

$$u(2,\theta) = 4\cos 2\theta \qquad \text{— (2pts).}$$

First, we get rid of 1: $u_0(r) = \frac{r^2}{4}$, $u = u_0(r) + v$ (5pts)

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0$$

$$v(2,\theta) = u(2,\theta) - \frac{r^2}{4}$$

$$= 4\cos 2\theta - 1$$

By the method of separation of variables. 5pts

$$v = a_0 + \sum_{n=1}^{+\infty} r^n (a_n\cos n\theta + b_n \sin n\theta)$$

$$= a_0 + a_2 r^2 \cos 2\theta$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} v = -1 \qquad \text{5pts}$$

$$a_2 = \frac{1}{\pi\cdot 4}\int_0^{2\pi} v\cos 2\phi = \frac{4}{4} = 1 \qquad \text{3pts}$$

So $v = -1 + r^2\cos 2\theta$ $\qquad u = \frac{r^2}{4} - 1 + r^2\cos 2\theta$.