Solutions to homework #7

Problem 1. (a) Use the method of separation of variables

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta - 2\pi\text{-periodic}$$

$$R'' + \frac{1}{r}R' - \frac{\lambda}{r^2}R = 0$$

So $\lambda_0 = 0 \Rightarrow R'' + \frac{1}{r}R' = 0 \Rightarrow R = c_1 + c_2 \log r.$

R bdd $\Rightarrow R = c_1$

$\lambda_n = n^2,\ n = 1, 2, \cdots \Rightarrow R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = 0$

$\Rightarrow R = c_1 r^n + c_2 r^{-n}$

R bdd for $r > 1 \Rightarrow R = c\, r^{-n}$

5pts

Thus

$$u(r,\theta) = a_0 + \sum_{n=1}^{+\infty} r^{-n}(a_n \cos n\theta + b_n \sin n\theta)$$

where $u(1,\theta) = a_0 + \sum_{n=1}^{+\infty}(a_n \cos n\theta + b_n \sin n\theta)$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)\, d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\cos n\theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\sin n\theta$$

5pts

Now $u(1,\theta) = \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ so

$a_0 = \frac{1}{2}, \quad a_2 = \frac{1}{2}, \quad a_n = 0,\ n \neq 0, 2, \quad b_n = 0,\ n = 1, 2 \ldots$

So $u = \frac{1}{2} + \frac{1}{2} r^{-2} \cos 2\theta$

(b) Since the boundary value is radially symmetric,

$u = u(r)$ so

$$u = c_1 + c_2 r^{-1}$$ 5 pts

$u(1) = 1 \Rightarrow c_1 + c_2 = 1$

$\lim_{r \to +\infty} u(r) = -2 \Rightarrow c_1 = -2$ 5 pts

So $c_1 = -2$, $c_2 = 3$

Problem 2: Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions

to
$$\begin{cases} u_t = k\Delta u, & x \in D, t > 0 \\ u(x,0) = \phi(x), & x \in D \\ \frac{\partial u}{\partial n} + a(x,t) u = g(x,t), & x \in \partial D \end{cases}$$

Set

$$v(x,t) = u_1(x,t) - u_2(x,t)$$ 2

Then v satisfies

$$v_t = k\Delta v$$

$$v(x,0) = u_1(x,0) - u_2(x,0) = \phi(x) - \phi(x) = 0, \quad x \in D$$

$$\frac{\partial v}{\partial n} + av = \frac{\partial u_1}{\partial n} + a u_1 - \left(\frac{\partial u_2}{\partial n} + a u_2\right) = g - g = 0, \quad x \in \partial D$$ 5 pts

Consider $E(t) = \frac{1}{2}\int_D v^2(x,t)\,dx$

It follows:

$$\frac{dE}{dt} = \frac{d}{dt}\int_D \frac{1}{2}v^2\,dx = \int_D v v_t\,dx$$

$$\overset{v_t = k\Delta v}{=\!=\!=} k\int_D v\Delta v\,dx$$

$$= k\int_D \left(\nabla(v\nabla v) - |\nabla v|^2\right)$$

$$\overset{\text{divergence}}{=\!=\!=} k\int_{\partial D} v\nabla v\cdot n - k\int_D |\nabla v|^2$$

5 pts

$$\underset{\frac{\partial v}{\partial n} = -av}{\overset{BC}{=\!=\!=}} -k\int_{\partial D} av^2 - k\int_D |\nabla v|^2$$

Since $a>0$, $\frac{dE}{dt} \le 0$. $E(t)$ is a decreasing function.

5 pts

5 So for $t>0$

$$E(t) \le E(0) = \frac{1}{2}\int_D v^2(x,0)\,dx = 0$$

$$\int_D v^2(x,t)\,dx = 0$$

5 pts

$\Rightarrow v(x,t) \equiv 0$.

$\Rightarrow u_1 = u_2$

So uniqueness is proved

Problem 3 : Step 1: $u = R(r)\,\Theta(\theta)$

$$R'' + \frac{1}{r}R' - \frac{\lambda}{r^2}R = 0, \quad R \text{ bdd}$$

$$\Theta'' + \lambda\Theta = 0, \quad \Theta'(0) = 0 = \Theta'\left(\frac{\pi}{4}\right)$$

2 pts

Step 2 : Solve (EVP).

$$\Theta'' + \lambda\Theta = 0, \quad \Theta'(0) = 0 = \Theta'\left(\frac{\pi}{4}\right)$$

$$\lambda_0 = 0, \quad \lambda_n = \left(\frac{n\pi}{\frac{\pi}{4}}\right)^2 = (4n)^2, \quad n = 1, 2, \cdots$$

$$\Theta_n(\theta) = \cos(4n\theta)$$

3 pts

$$\lambda_0 = 0, \quad R'' + \frac{1}{r}R' = 0 \Rightarrow R = c_1 + c_2 \log r$$

Since $1 < r < 2$, we keep both terms

$$\lambda_n = (4n)^2, \quad R'' + \frac{1}{r}R' - \frac{(4n)^2}{r^2}R = 0 \Rightarrow R = c_1 r^{4n} + c_2 r^{-4n}$$

Since $1 < r < 2$, we have to keep both terms

5 p

Step 3. Sum-up

$$u(r,\theta) = a_0 + b_0 \log r + \sum_{n=1}^{+\infty} \left(a_n r^{4n} + b_n r^{-4n}\right)\cos(4n\theta)$$

2 p

BC: $u(1,\theta) = -\cos 4\theta$

$$\Rightarrow \quad a_0 + b_0 \log 1 = 0 \qquad (1)$$

$$a_1 1^{4} + b_1 \cdot 1^{-4} = -1 \qquad (2)$$

$$a_n + b_n = 0, \quad n > 1 \qquad (3)$$

3 pts

$u(2,\theta)=1 \Rightarrow$

$$a_0 + b_0 \log 2 = 1 \quad -(4)$$

$$a_n 2^{4n} + b_n 2^{-4n} = 0, \quad n \geq 1 \quad (5)$$

2 pts

From (1) + (4): $a_0 = 0, \quad b_0 = \frac{1}{\log 2}$

(2) + (5): $\left.\begin{aligned} a_1 + b_1 &= -1 \\ a_1 2^4 + b_1 2^{-4} &= 0 \end{aligned}\right\} \Rightarrow a_1 = \frac{1}{2^8 - 1}, \; b_1 = \frac{2^8}{1-2^8}$

(3) + (5): $a_n = 0, \; b_n = 0, \; n \geq 2$

3 pts

Thus

$$u(r,\theta) = \frac{\log r}{\log 2} + \left(\frac{1}{2^8-1} r^4 + \frac{2^8}{1-2^8} r^{-4}\right) \cos 4\theta.$$

Problem 4: By the method of separation of variables:

$u = R(r)\,T(t)$

$R'' + \frac{1}{r}R + \lambda R = 0, \quad R(a) = 0$

$T'' + c^2 \lambda_n T = 0$

5 pts

So $R = J(\sqrt{\lambda_n}\, r), \quad \lambda_n = \left(\frac{z_n}{a}\right)^2$, z_n - n-th zero of Bessel function $J(x)$

$T = c_1 \cos(c\sqrt{\lambda_n}\, t) + c_2 \sin(c\sqrt{\lambda_n}\, t)$

5 pts

Sum-up

$$u(r,t) = \sum_{n=1}^{+\infty} \left(a_n \cos(c\sqrt{\lambda_n}\, t) + b_n \sin(c\sqrt{\lambda_n}\, t) \right) J(\sqrt{\lambda_n}\, r)$$

2pts

$u(r,0) = \phi(r) \Rightarrow$

$$\sum_{n=1}^{+\infty} a_n J(\sqrt{\lambda_n}\, r) = \phi(r)$$

$$a_n = \frac{\int_0^a r J(\sqrt{\lambda_n}\, r) \phi(r)\, dr}{\int_0^a r \left(J(\sqrt{\lambda_n}\, r)\right)^2 dr}$$

4pts

$u_t(r,0) = \psi(r) \Rightarrow$

$$\sum_{n=1}^{+\infty} c\sqrt{\lambda_n}\, b_n J(\sqrt{\lambda_n}\, r) = \psi(r)$$

$$b_n = \frac{\int_0^a r J(\sqrt{\lambda_n}\, r) \phi(r)\, dr}{c\sqrt{\lambda_n} \int_0^a r \left(J(\sqrt{\lambda_n}\, r)\right)^2 dr}$$

4pts

Problem5: Step1. $u = R(r) Z(z) T(t)$

$$\frac{T'}{kT} = \frac{R'' + \frac{1}{r} R'}{R} + \frac{Z''}{Z}$$

5pts

Let $\dfrac{R'' + \frac{1}{r} R'}{R} = -\lambda \Rightarrow R'' + \frac{1}{r} R' + \lambda R = 0,\ R(a) = 0$

$\dfrac{Z''}{Z} = -\mu \Rightarrow Z'' + \mu Z = 0,\ Z'(0) = Z'(b) = 0$

$T' + k(\lambda + \mu) T = 0.$

Step2: Solve the two EVPs.

$$\lambda_n = \left(\frac{z_n}{a}\right)^2, \quad R_n = J\left(\frac{z_n}{a} r\right)$$

$$\mu_m = \left(\frac{m\pi}{b}\right)^2, m = 0, 1, 2, \cdots \quad Z_m = \cos\left(\frac{m\pi}{b} z\right)$$

6 pts

$$T' + k(\lambda_n + \mu_m) T = 0$$

$$T = c \; e^{-k(\lambda_n + \mu_m)t}$$

Step3. Sum up

$$u(r, z, t) = \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} a_{m,n} \, e^{-k\left(\left(\frac{z_n}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right)t} J\left(\frac{z_n}{a} r\right) \cos\left(\frac{m\pi}{b} z\right).$$

(3pts)

$$u(r, z, 0) = \phi(r, z)$$

$$\Rightarrow \phi(r, z) = \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} a_{m,n} \, e^{\cancel{-k(\lambda_n + \mu_m)t}} J(\sqrt{\lambda_n} r) \cos(\mu_m z)$$

Fix r:

$$\sum_{n=1}^{+\infty} a_{m,n} \, e^{\cancel{-k(\lambda_n + \mu_m)t}} J(\sqrt{\lambda_n} r) = \frac{\int_0^b \phi(r, z) \cos(\mu_m z) \, dz}{\int_0^b \cos^2(\mu_m z) \, dz}$$

$$= \frac{2}{b} \cdot \int_0^b \phi(r, z) \cos(\mu_m z) \, dz$$

So $$a_{m,n} = \frac{2\int_0^a\left(\int_0^b \phi(r,z)\, r J(\sqrt{\lambda_n}\, r) \cos \mu_m z \, dz\right)dr}{b\int_0^a r\left(J(\sqrt{\lambda_n}\, r)\right)^2 dr}$$

$\longrightarrow$ (6 pts)

Pre